UNIT 1 2-D and 3-D TRANSFORMATIONS

Structure **Page Nos.**

1.0 INTRODUCTION

In the previous Block, we have presented approaches for the generation of lines and polygonal regions. We know that once the objects are created, the different applications may require variations in these. For example, suppose we have created the scene of a room. As we move along the room we find the object's position comes closer to us, it appears bigger even as its orientation changes. Thus we need to alter or manipulate these objects. Essentially this process is carried out by means of transformations. Transformation is a process of changing the position of the object or maybe any combination of these.

The objects are referenced by their coordinates. Changes in orientation, size and shape are accomplished with **geometric transformations** that allow us to calculate the new coordinates. The basic geometric transformations are translation, rotation, scaling and shearing. The other transformations that are often applied to objects include reflection.

In this Block, we will present transformations to manipulate these geometric 2-D objects through Translation, and Rotation on the screen. We may like to modify their shapes either by magnifying or reducing their sizes by means of Scaling transformation. We can also find similar but new shapes by taking mirror reflection with respect to a chosen axis of references. Finally, we extend the 2-D transformations to 3-D cases.

1.1 OBJECTIVES

After going through this unit, you should be able to:

- describe the basic transformations for 2-D translation, rotation, scaling and shearing;
- discuss the role of composite transformations;

- describe composite transformations for Rotation about a point and reflection about a line;
- define and explain the use of homogeneous coordinate systems for the transformations, and
- extend the 2-D transformations discussed in the unit to 3-D transformations.

1.2 BASIC TRANSFORMATIONS

Consider the xy-coordinate system on a plane. An object (say *Obj*) in a plane can be considered as a set of points. Every object point P has coordinates (x,y), so the object is the sum total of all its coordinate points (see *Figure 1*). Let the object be moved to a new position. All the coordinate points P'(x',y') of a new object *Obj'* can be obtained from the original points P(x,y) by the application of a geometric transformation.

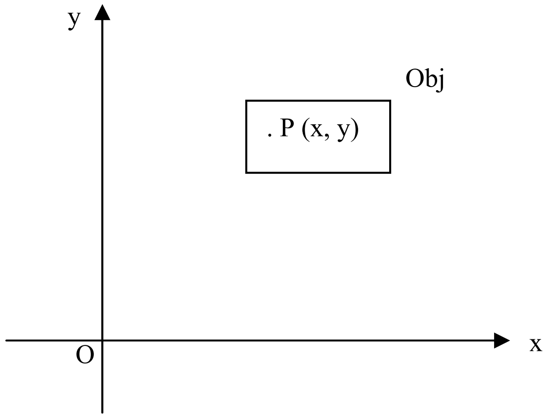


Figure 1

1.2.1 Translation

Translation is the process of changing the position of an object. Let an object point P(x,y)=xI+yJ be moved to P'(x',y') by the given translation vector V= t_xI + t_yJ, where t_x and t_y is the translation factor in x and y directions, such that

P'=P+V. ----------------(1)

In component form, we have

$$Tv=\begin{cases} x'=x+t_x \text{ and} \\ y'=y+t_y \end{cases} \qquad \text{----------------(2)}$$

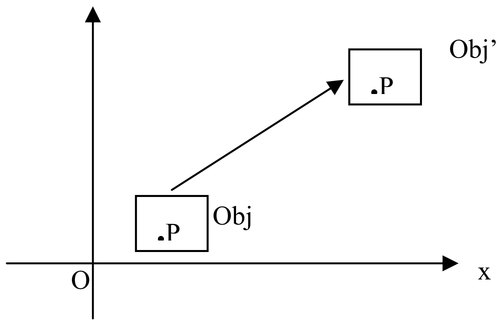


Figure 2

As shown in *Figure 2*, P' is the new location of P, after moving t_x along x-axis and t_y along y-axis. It is not possible to develop a relation of the form.

$P'=P.T_v$ ---------------(3)

Where Tv is the transformation for translation **in matrix form**.

That is, we cannot represent the translation transformation in (2x2) matrix form (2-D Euclidean system).

Any transformation operation can be represented as a (2x2) matrix form, except translation, i.e., translation transformation cannot be expressed as a (2x2) matrix form (2-D Euclidean system). But by using Homogeneous coordinate system (HCS), we can represent translation transformation in matrix form. The HCS and advantages of using HCS is discussed, in detail, in section 1.4.

Relation between 2-D Euclidean (Cartesian) system and HCS

Let P(x,y) be any point in 2-D Euclidean system. In Homogeneous Coordinate system, we add a third coordinate to the point. Instead of (x,y), each point is represented by a triple (x,y,H) such that H≠0; with the condition that (x1,y1,H1)=(x2,y2,H2) ↔ x1/H1 = x2/H2 ; y1/H1 = y2/H2. In two dimensions the value of H is usually kept at 1 for simplicity. (If we take H=0 here, then this represents point at infinity, i.e, generation of horizons).

The following table shows a relationship between 2-D Euclidean (Cartesian coordinate) system and HCS.

2-D Euclidian System	Homogeneous Coordinate System (HCS)
Any point (x,y) ⟶	(x,y,1)
H≠0); (x/H,y/H) ⟵	If (x,y,H) be any point in HCS(such that then (x,y,H)=(x/H,y/H,1), i.e. (x,y,H)

For translation transformation, any point (x,y)→(x+tx,y+ty) in 2-D Euclidian system. Using HCS, this translation transformation can be represented as (x,y,1) → (x+tx,y+ty,1). In two dimensions the value of H is usually kept at 1 for simplicity. Now, we are able to represent this translation transformation in matrix form as:

$$(x',y',1)=(x,y,1)\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ tx & ty & 1\end{pmatrix}$$

$P'_h=P_h.T_v$ ----------------(4)

Where P'_h and P_h represents object points in Homogeneous Coordinates and Tv is called homogeneous transformation matrix for translation. Thus, P'_h, the new coordinates of a transformed object, can be found by multiplying previous object coordinate matrix, P_h, with the transformation matrix for translation Tv.

The ***advantage*** of introducing the matrix form of translation is that it simplifies the operations on complex objects i.e., we can now build complex transformations by multiplying the basic matrix transformations. This process is called *concatenation of* *matrices* and the resulting matrix is often referred as the *composite transformation* matrix.

We can represent the basic transformations such as rotation, scaling shearing, etc., as 3x3 homogeneous coordinate matrices to make matrix multiplication compatibility with the matrix of translation. This is accomplished by augmenting the 2x2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with a third column $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and a third row (0,0,1). That is

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus, the new coordinates of a transformed object can be found by multiplying previous object coordinate matrix with the required transformation matrix. That is

$$\begin{bmatrix} \text{New Object} \\ \text{Coordinate} \\ \text{matrix} \end{bmatrix} = \begin{bmatrix} \text{Previous object} \\ \text{Coordinate} \\ \text{matrix} \end{bmatrix} . \begin{bmatrix} \text{Transformation} \\ \text{matrix} \end{bmatrix}$$

Example1: Translate a square ABCD with the coordinates A(0,0),B(5,0),C(5,5),D(0,5) by 2 units in x-direction and 3 units in y-direction.

Solution: We can represent the given square, in matrix form, using homogeneous coordinates of vertices as:

$$\begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{pmatrix} x1 & y1 & 1 \\ x2 & y2 & 1 \\ x3 & y3 & 1 \\ x4 & y4 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 5 & 0 & 1 \\ 5 & 5 & 1 \\ 0 & 5 & 1 \end{pmatrix}$$

The translation factors are, tx=2, ty=3

The transformation matrix for translation :

$$T_v = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ tx & ty & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

New object point coordinates are:

$$[A'B'C'D'] = [ABCD].T_v$$

$$\begin{array}{c} A' \\ B' \\ C' \\ D' \end{array} \begin{pmatrix} x'_1 & y'_1 & 1 \\ x'_2 & y'_2 & 1 \\ x'_3 & y'_3 & 1 \\ x'_4 & y'_4 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 5 & 0 & 1 \\ 5 & 5 & 1 \\ 0 & 5 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix}$$

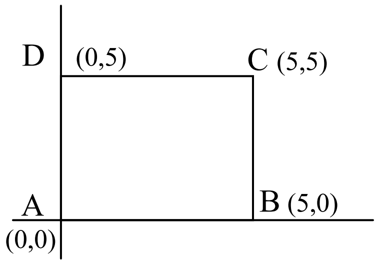
$$= \begin{pmatrix} 2 & 3 & 1 \\ 7 & 3 & 1 \\ 7 & 8 & 1 \\ 2 & 8 & 1 \end{pmatrix}$$

Thus, $A'(x'_1,y'_1)=(2,3)$

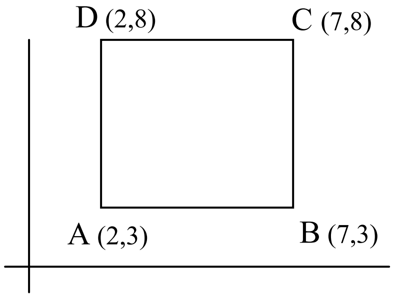
$B'(x'_2,y'_2)=(7,3)$

$C'(x'_3,y'_3)=(7,8)$ and $D'(x'_4,y'_4)=(2,8)$

The graphical representation is given below:



a) Square before Translation



b) Square after Translation

1.2.2 Rotation

In 2-D rotation, an object is rotated by an angle θ with respect to the origin. This angle is assumed to be positive for anticlockwise rotation. There are two cases for 2-D rotation, *case1*- rotation about the origin and *case2* rotation about an arbitrary point. If, the rotation is made about an arbitrary point, a set of basic transformation, i.e., composite transformation is required. For 3-D rotation involving 3-D objects, we need to specify both the angle of rotation and the axis of rotation, about which rotation has to be made. Here, we will consider case1 and in the next section we will consider *case2*.

Before starting case-1 or case-2 you must know the relationship between **polar coordinate system** and **Cartesian system:**

Relation between **polar coordinate system** and **Cartesian system**

A frequently used non-cartesian system is Polar coordinate system. The following *Figure A* shows a polar coordinate reference frame. In polar coordinate system a coordinate position is specified by r and θ, where r is a radial distance from the coordinate origin and θ is an angular displacements from the horizontal (see *Figure 2A*). Positive angular displacements are counter clockwise. Angle θ is measured in degrees. One complete counter-clockwise revolution about the origin is treated as 360^0. A relation between Cartesian and polar coordinate system is shown in *Figure 2B*.

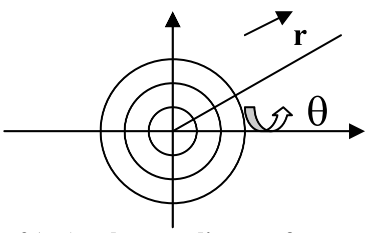


Figure 2A: A polar coordinate reference-frame

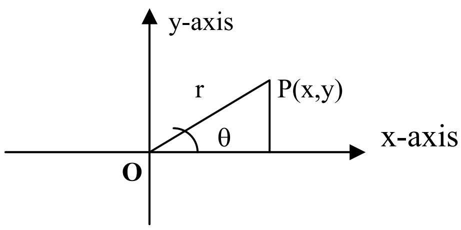


Figure 2B: Relation between Polar and Cartesian coordinates

Consider a right angle triangle in *Figure B*. Using the definition of trigonometric functions, we transform polar coordinates to Cartesian coordinates as:

$x=r.\cos\theta$
$y=r.\sin\theta$

The inverse transformation from Cartesian to Polar coordinates is:

$r=\sqrt{(x^2+y^2)}$ and $\theta=\tan^{-1}(y/x)$

Case 1: Rotation about the origin

Given a 2-D point P(x,y), which we want to rotate, with respect to the origin O. The vector OP has a length 'r' and making a positive (anticlockwise) angle ϕ with respect to x-axis.

Let P'(x'y') be the result of rotation of point P by an angle θ about the origin, which is shown in *Figure 3*.

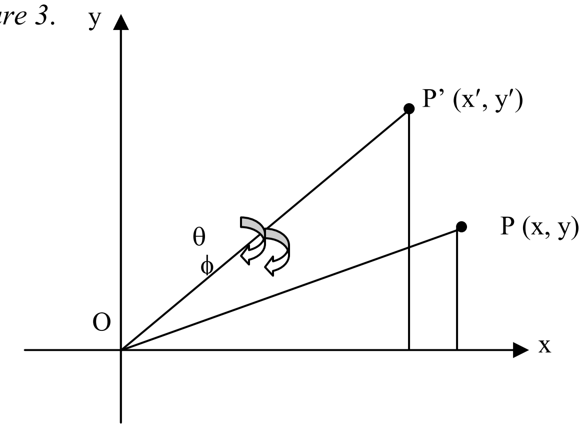


Figure 3

$P(x,y) = P(r.\cos\phi, r.\sin\phi)$
$P'(x',y') = P[r.\cos(\phi+\theta), r\sin(\phi+\theta)]$

The coordinates of P' are:

$x' = r.\cos(\theta+\phi) = r(\cos\theta\cos\phi - \sin\theta\sin\phi)$
$\quad = x.\cos\theta - y.\sin\theta$ (where $x = r\cos\phi$ and $y = r\sin\phi$)

similarly;

$y' = r\sin(\theta+\phi) = r(\sin\theta\cos\phi + \cos\theta.\sin\phi)$
$\quad = x\sin\theta + y\cos\theta$

Thus,

$$R_\theta = \left\{ \begin{array}{l} x' = x.\cos\theta - y.\sin\theta \\ y' = x\sin\theta + y\cos\theta \end{array} \right\} = R_\theta$$

Thus, we have obtained the new coordinate of point P after the rotation. In matrix form, the transformation relation between P' and P is given by:

$$(x'y') = (x,y) \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

that is $P' = P.R_\theta$ ---------(5)

where P' and P represent object points in 2-D Euclidean system and R_θ is transformation matrix for **anti-clockwise** Rotation.

In terms of HCS, equation (5) becomes

$$(x', y', 1) = (x, y, 1) \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{---------(6)}$$

That is $P'_h = P_h.R_\theta$, ---------(7)

Where P'_h and P_h represents object points, after and before required transformation, in Homogeneous Coordinates and R_θ is called homogeneous transformation matrix for **anticlockwise** Rotation. Thus, P'_h, the new coordinates of a transformed object, can be found by multiplying previous object coordinate matrix, P_h, with the transformation matrix for Rotation R_θ.

Note that for **clockwise** rotation we have to put $\theta = -\theta$, thus the rotation matrix R_θ, in HCS, becomes

$$R_{-\theta} = \begin{pmatrix} \cos(-\theta) & \sin(-\theta) & 0 \\ -\sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 2: Perform a 45^0 rotation of a triangle A(0,0),B(1,1),C(5,2) about the origin.

Solution: We can represent the given triangle, in matrix form, using homogeneous coordinates of the vertices:

$$[ABC] = \begin{pmatrix} A & 0 & 0 & 1 \\ B & 1 & 1 & 1 \\ C & 5 & 2 & 1 \end{pmatrix}$$

The matrix of rotation is: $R_\theta = R_{45^0} = \begin{pmatrix} \cos 45^0 & \sin 45^0 & 0 \\ -\sin 45^0 & \cos 45^0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

So the new coordinates A'B'C' of the rotated triangle ABC can be found as:

$$[A'B'C'] = [ABC].\,R_{45^\circ} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2/2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \sqrt{2} & 1 \\ 3\sqrt{2}/2 & 7\sqrt{2}/2 & 1 \end{bmatrix}$$

Thus $A'=(0,0)$, $B'=(0,\sqrt{2})$, $C'=(3\sqrt{2}/2, 7\sqrt{2}/2)$

The following *Figure (a)* shows the original, triangle [ABC] and *Figure (b)* shows triangle after the rotation.

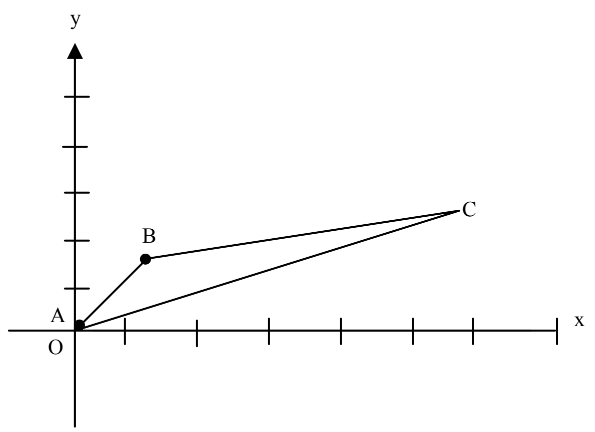


Figure (a)

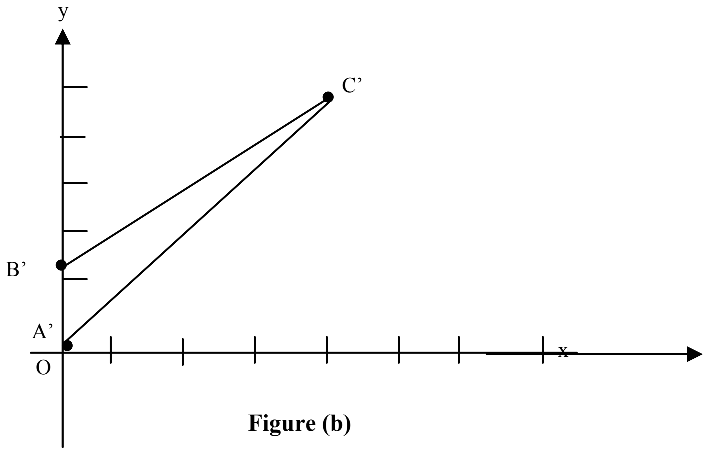


Figure (b)

☞ Check Your Progress 1

1) What are the basic advantages of using Homogeneous coordinates system.

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2) A square consists of vertices A(0,0),B(0,1),C(1,1),D(1,0). After the translation C is found to be at the new location (6,7). Determine the new location of other vertices.

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3) A point P(3,3) makes a rotating of 45^0 about the origin and then translating in the direction of vector v=5I+6J. Find the new location of P.

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4) Find the relationship between the rotations R_θ, $R_{-\theta}$, and R_θ^{-1} .

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1.2.3 Scaling

Scaling is the process of expanding or compressing the dimensions (i.e., size) of an object. An important application of scaling is in the development of viewing transformation, which is a mapping from a window used to clip the scene to a view port for displaying the clipped scene on the screen.

Let P(x,y) be any point of a given object and s_x and s_y be scaling factors in x and y directions respectively, then the coordinate of the scaled object can be obtained as:

$$\left.\begin{aligned} x' &= x.s_x \\ y' &= y.s_y \end{aligned}\right\} \qquad \text{--------(8)}$$

If the scale factor is 0<s<1, then it reduces the size of an object and if it is more then 1, it magnifies the size of the object along an axis.

For example, assume $s_x > 1$.

i) Consider (x,y)→(2.x,y) i.e., Magnification in x-direction with scale factor $s_x = 2$.

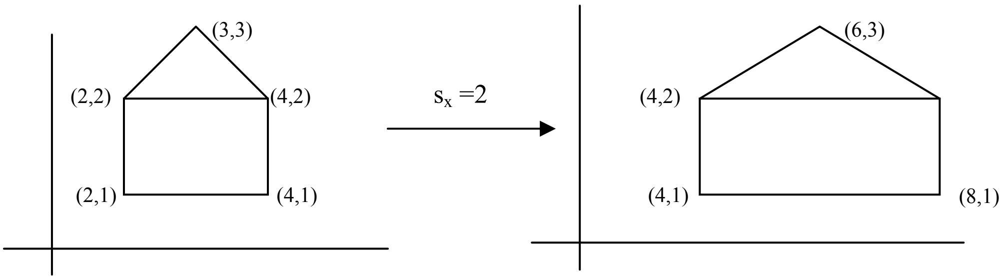

Figure a): Object before Scaling **Figure b): Object after Scaling with $s_x = 2$**

ii) Similarly, assume $s_y > 1$ and consider (x,y)→(x,2.y), i.e., Magnification in y-direction with scale factor $s_y = 2$.

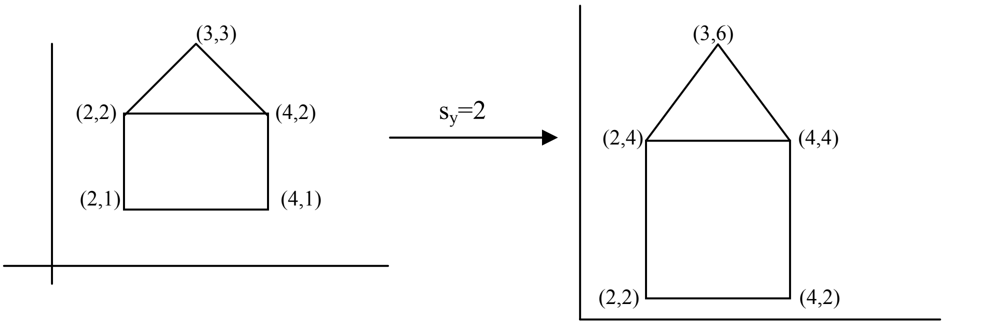

Figure a): Object before Scaling **Figure b): Object after Scaling with Sy=2**

iii) Consider (x,y)→($x.s_x$,y) where $0 < s_x = y_2 < 1$ i.e., Compression in x-direction with scale factor $s_x = 1/2$.

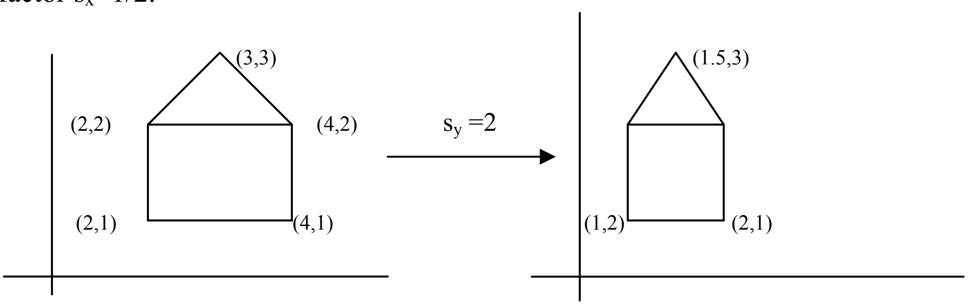

Figure a): Object before Scaling **Figure b): Object after Scaling with Sx=1/2**

Thus, the general scaling is (x,y)→ ($x.s_x$,$y.s_y$) i.e., magnifying or compression in both x and y directions depending on Scale factors s_x and s_y. We can represent this in matrix form (2-D Euclidean system) as:

$$(x',y') = (x,y) \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \qquad \text{----(9)}$$

In terms of HCS, equation (9) becomes:

$$(x',y',1)=(x,y,1)\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{----(10)}$$

that is $P'_h=P_h.s_{sx,sy}$ ----(11)

Where P_h and P'_h represents object points, before and after required transformation, in Homogeneous Coordinates and $s_{sx,sy}$ is called transformation matrix for general scaling with scaling factor s_x and s_y .

Thus, we have seen any positive value can be assigned to scale factors s_x and s_y. We have the following three cases for scaling:

Case 1: If the values of s_x and s_y are less then 1, then the size of the object will be reduced.

Case2: If both s_x and s_y are greater then 1, then the size of the object is enlarged.

Case3: If we have the same scaling factor (i.e. s_x=s_x=S), then there will be uniform scaling (either enlargement or compression depending on the value of S_x and Sy) in both x and y directions.

Example3: Find the new coordinates of a triangle A(0,0),B(1,1),C(5,2) after it has been (a) magnified to twice its size and (b) reduced to half its size.

Solution: Magnification and reduction can be achieved by a uniform scaling of s units in both the x and y directions. If, s>1, the scaling produces magnification. If, s<1, the result is a reduction. The transformation can be written as: (x,y,1)→(s.x,s.y,1). In matrix form this becomes

$$(x,y,1).\begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{pmatrix} = (s.x,s.y,1)$$

We can represent the given triangle, shown in *Figure (a)*, in matrix form, using homogeneous coordinates of the vertices as :

$$\begin{pmatrix} A & 0 & 0 & 1 \\ B & 1 & 1 & 1 \\ C & 5 & 2 & 1 \end{pmatrix}$$

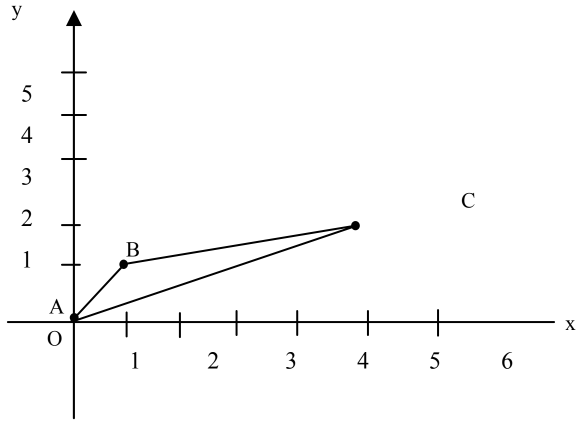

Figure a: Object before scaling

(a) choosing s=2

The matrix of scaling is: $S_{s_x,s_y} = S_{2,2} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So the new coordinates A'B'C' of the scaled triangle ABC can be found as:

$$[A'B'C'] = [ABC].\ R_{2,2} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 2 & 1 \\ 10 & 4 & 1 \end{bmatrix}$$

Thus, A'=(0,0), B'=(2,2), C'= (10, 4)

(b) Similarly, here, s=1/2 and the new coordinates are A''=(0,0), B''=(1/2,1/2), C''=(5/2,1). The following figure (b) shows the effect of scaling with $s_x=s_y=2$ and (c) with $s_x=s_y=s=1/2$.

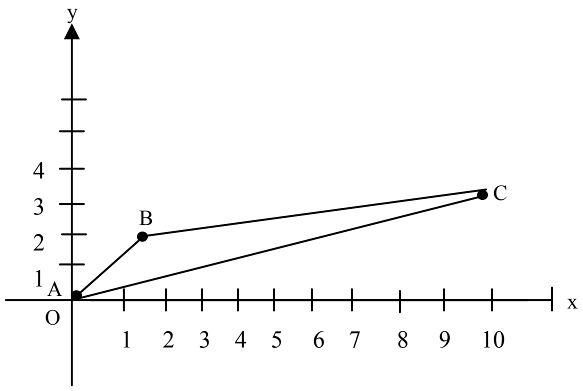

Figure b: Object after scaling with Sx = Sy = 2

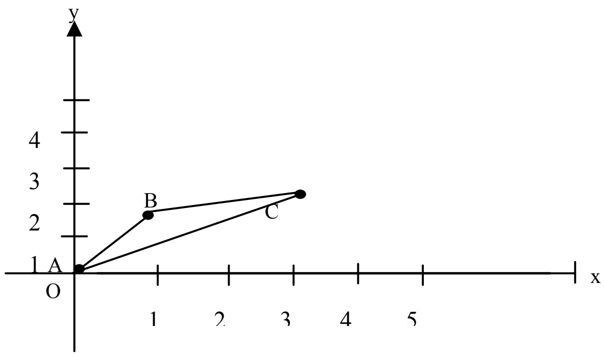

Figure c: Object after scaling with $S_x = S_y = 1/2$

1.2.4 Shearing

Shearing transformations are used for modifying the shapes of 2-D or 3-D objects. The effect of a shear transformation looks like "pushing" a geometric object in a direction that is parallel to a coordinate plane (3D) or a coordinate axis (2D). How far a direction is pushed is determined by its *shearing factor.*

One familiar example of shear is that observed when the top of a book is moved relative to the bottom which is fixed on the table.

In case of 2-D shearing, we have two types namely *x-shear* and *y-shear*.

In *x-shear*, one can push in the *x*-direction, positive or negative, and keep the *y*-direction unchanged, while in *y-shear*, one can push in the *y*-direction and keep the *x*-direction fixed.

x-shear about the origin

Let an object point P(x,y) be moved to P'(x',y') in the x-direction, by the given scale parameter 'a',i.e., P'(x'y') be the result of x-shear of point P(x,y) by scale factor a about the origin, which is shown in *Figure 4.*

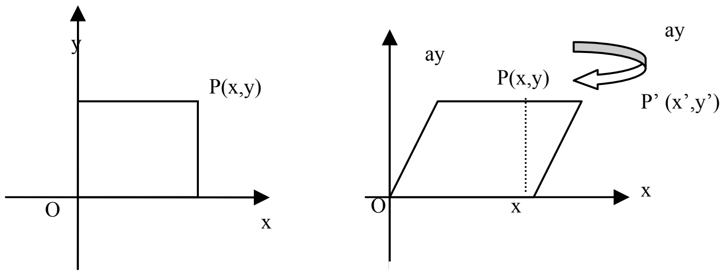

Figure 4

Thus, the points P(x,y) and P'(x',y') have the following relationship:

$$\left.\begin{aligned} x' &= x + ay \\ y' &= y \end{aligned}\right\} = Sh_x(a) \qquad \text{------(11a)}$$

where 'a' is a constant (known as shear parameter) that measures the degree of shearing. If a is negative then the shearing is in the opposite direction.

Note that P(0,H) is taken into P'(aH,H). It follows that the shearing angle A (the angle through which the vertical edge was sheared) is given by:

tan(A) = aH/H = a.

So the parameter a is just the tan of the shearing angle. In matrix form (2-D Euclidean system), we have

$$(x',y')=(x,y)\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \qquad \text{-------(12)}$$

In terms of Homogeneous Coordinates, equation (12) becomes

$$(x',y',1)=(x,y,1).\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{-------(13)}$$

That is, $P'_h = P_h\, Sh_x(a)$ --------(14)

Where P_h and P'_h represents object points, before and after required transformation, in Homogeneous Coordinates and $Sh_x(a)$ is called homogeneous transformation matrix for x-shear with scale parameter 'a' in the x-direction.

y-shear about the origin

Let an object point P(x,y) be moved to P'(x',y') in the x-direction, by the given scale parameter 'b'. i.e., P'(x'y') be the result of y-shear of point P(x,y) by scale factor 'b' about the origin, which is shown in *Figure 5(a)*.

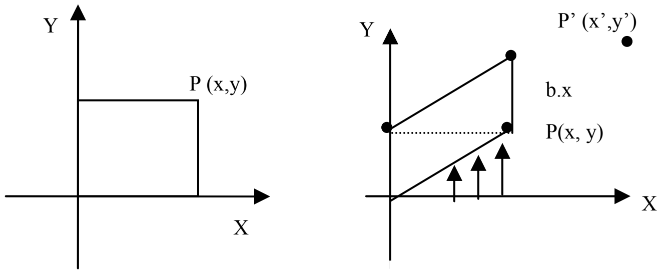

Figure 5 (a)

Thus, the points P(x,y) and P'(x',y') have the following relationship :

$$\left.\begin{aligned} x' &= x \\ y' &= y+bx \end{aligned}\right\} = Sh_y(b) \qquad \text{----------(15)}$$

where 'b' is a constant (known as shear parameter) that measures the degree of shearing. In matrix form, we have

$$(x',y')=(x,y)\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \qquad \text{----------(16)}$$

In terms of Homogeneous Coordinates, equation (16) becomes

$$(x',y',1)=(x,y,1)\begin{bmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{---------(17)}$$

That is, $P'_h = P_h.Sh_y(b)$ ---------(18)

Where P_h and P'_h represents object points, before and after required transformation, in Homogeneous Coordinates and $Sh_y(b)$ is called homogeneous transformation matrix for y-shear with scale factor 'b' in the y-direction.

xy-shear about the origin

Let an object point P(x,y) be moved to P'(x',y') as a result of shear transformation in both *x*- and *y*-directions with shearing factors *a* and *b*, respectively, as shown in *Figure 5(b)*.

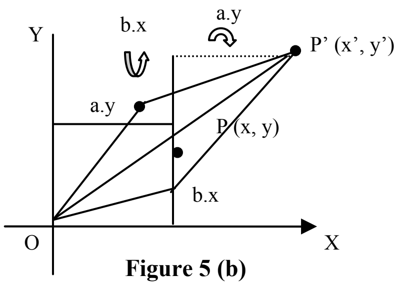

Figure 5 (b)

The points P(x,y) and P'(x',y') have the following relationship :

$$\left.\begin{aligned} x' &= x+ay \\ y' &= y+bx \end{aligned}\right\} = Sh_{xy}(a,b) \qquad \text{----------(19)}$$

where 'ay' and 'bx' are shear factors in x and y directions, respectively. The xy-shear is also called *simultaneous shearing* or *shearing* for short.

In matrix form, we have,

$$(x',y')=(x,y)\begin{bmatrix} 1 & b \\ a & 1 \end{bmatrix} \qquad \text{---------(20)}$$

In terms of Homogeneous Coordinates, we have

$$(x',y',1)=(x,y,1)\begin{bmatrix} 1 & b & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{---------(21)}$$

That is, $P'_h = P_h.Sh_{xy}(a,b)$ ----------(22)

Where P_h and P'_h represent object points, before and after required transformation, in Homogeneous Coordinates and $Sh_{xy}(a,b)$ is called homogeneous transformation matrix for xy-shear in both *x*- and *y*-directions with shearing factors *a* and *b*, respectively,

Special case: when we put b=0 in equation (21), we have *shearing in x-direction*, and when a=0, we have *Shearing in the y-direction,* respectively**.**

Example 4: A square ABCD is given with vertices A(0,0),B(1,0),C(1,1), and D(0,1). Illustrate the effect of a) x-shear b) y-shear c) xy-shear on the given square, when a=2 and b=3.

Solution: We can represent the given square ABCD, in matrix form, using homogeneous coordinates of vertices as:

$$\begin{pmatrix} A & 0 & 0 & 1 \\ B & 1 & 0 & 1 \\ C & 1 & 1 & 1 \\ D & 0 & 1 & 1 \end{pmatrix}$$

a) The matrix of x-shear is:

$$Sh_x(a) = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So the new coordinates A'B'C'D' of the x-sheared object ABCD can be found as:
[A'B'C'D']=[ABCD]. $Sh_x(a)$

$$[A'B'C'D'] = \begin{pmatrix} A & 0 & 0 & 1 \\ B & 1 & 0 & 1 \\ C & 1 & 1 & 1 \\ D & 0 & 1 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 3 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

Thus, A'=(0,0), B'=(1,0), C'=(3,1) and D'=(2,1).

b) Similarly the effect of shearing in the y direction can be found as:
[A'B'C'D']=[ABCD].$Sh_y(b)$

$$[A'B'C'D'] = \begin{pmatrix} A & 0 & 0 & 1 \\ B & 1 & 0 & 1 \\ C & 1 & 1 & 1 \\ D & 0 & 1 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 3 & 1 \\ 1 & 4 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Thus, A'=(0,0), B'=(1,3), C'=(1,4) and D'=(0,1).

c) Finally the effect of shearing in both directions can be found as:
[A'B'C'D']=[ABCD]. $Sh_{xy}(a,b)$

$$[A'B'C'D'] = \begin{pmatrix} A & 0 & 0 & 1 \\ B & 1 & 0 & 1 \\ C & 1 & 1 & 1 \\ D & 0 & 1 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 3 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 3 & 1 \\ 3 & 4 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

Thus, A'=(0,0), B'=(1,3), C'=(3,4) and D'=(2,1).

Figure (a) shows the original square, *figure (b)-(d)* shows shearing in the x, y and both directions respectively.

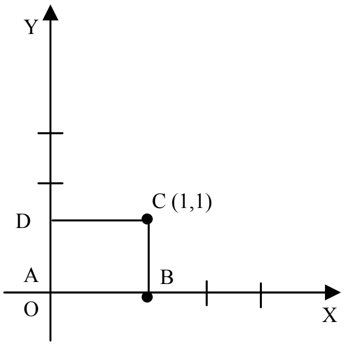

Figure (a)

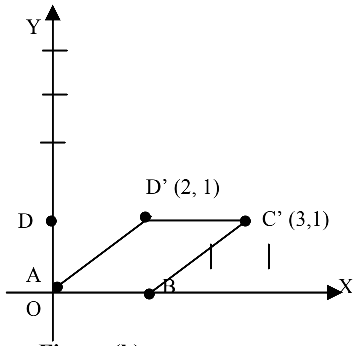

Figure (b)

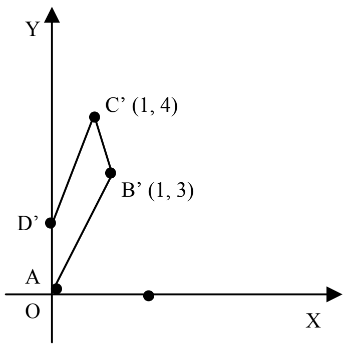

Figure (c)

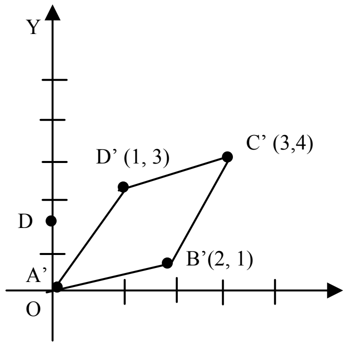

Figure (d)

Example 5: What is the use of Inverse transformation? Give the Inverse transformation for translation, rotation, reflection, scaling, and shearing.

Solution: We have seen the basic matrix transformations for translation, rotation, reflection, scaling and shearing with respect to the origin of the coordinate system. By multiplying these basic matrix transformations, we can build complex transformations, such as rotation about an arbitrary point, mirror reflection about a line etc. This process is called *concatenation of matrices* and the resulting matrix is often referred to as the *composite transformation matrix*. Inverse transformations play an important role when you are dealing with composite transformation. They come to the rescue of basic transformations by making them applicable during the construction of composite transformation. You can observed that the Inverse transformations for translation, rotation, reflection, scaling and shearing have the following relations, and v, θ, a, b, sx, sy, sz are all parameter involved in the transformations.

1) $T_v^{-1} = T_{-v}$
2) $R_\theta^{-1} = R_{-\theta}$
3) (i) $Sh_x^{-1}(a) = Sh_x(-a)$
   (ii) $Sh_y^{-1}(b) = Sh_x(-b)$
   (iii) $Sh_{xy}^{-1}(a,b) = Sh_x(-a,-b)$
4) $S^{-1}_{sx,sy,sz} = S_{1/sx,1/sy,1/sz}$
5) The transformation for mirror reflection about principal axes do not change after inversion.
   (i) $Mx^{-1} = M_{-x} = M_x$
   (ii) $My^{-1} = M_{-y} = M_y$
   (iii) $Mz^{-1} = M_{-z} = M_z$,

6) The transformation for rotations made about x,y,z axes have the following inverse:
   (i) $R^{-1}_{x,\theta} = R_{x,-\theta} = R^{T}_{x,\theta}$
   (ii) $R^{-1}_{y,\theta} = R_{y,-\theta} = R^{T}_{y,\theta}$
   (iii) $R^{-1}_{z,\theta} = R_{z,-\theta} = R^{T}_{z,\theta}$

☞ **Check Your Progress 2**

1) Differentiate between the Scaling and Shearing transformation.

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2) Show that $S_{a,b} .S_{c,d} = S_{c,d} .S_{a,b} = S_{ac,bd}$

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3) Find the 3x3 homogeneous co-ordinate transformation matrix for each of the following:
   a) Shift an image to the right by 3 units.
   b) Shift the image up by 2 units and down 1 units.
   c) Move the image down 2/3 units and left 4 units.

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4) Find the condition under which we have $S_{s_x,s_y} .R_\theta = R_\theta. S_{s_x,s_y}$.

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5) Is a simultaneous shearing the same as the shearing in one direction followed by a shearing in another direction? Why?

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1.3 COMPOSITE TRANSFORMATIONS

We can build complex transformations such as rotation about an arbitrary point, mirror reflection about a line, etc., by multiplying the basic matrix transformations. This process is called *concatenation of matrices* and the resulting matrix is often referred to as the *composite transformation matrix*. In composite transformation, a previous transformation is pre-multiplied with the next one.

In other words we can say that a sequence of the transformation matrices can be concatenated into a single matrix. This is an effective procedure as it reduces because instead of applying initial coordinate position of an object to each transformation matrix, we can obtain the final transformed position of an object by applying composite matrix to the initial coordinate position of an object. In other words we can say that a sequence of transformation matrix can be concatenated matrix into a single matrix. This is an effective procedure as it reduces computation because instead of applying initial coordinate position of an object to each transformation matrix, we can obtain the final transformed position of an object by applying composite matrix to the initial coordinate position of an object.

1.3.1 Rotation about a Point

Given a 2-D point P(x,y), which we want to rotate, with respect to an arbitrary point A(h,k). Let P'(x'y') be the result of anticlockwise rotation of point P by angle θ about A, which is shown in *Figure 6*.

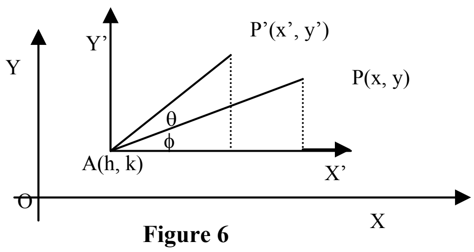


Figure 6

Since, the rotation matrix R_θ is defined only with respect to the origin, we need a set of basic transformations, which constitutes the composite transformation to compute the rotation about a given arbitrary point A, denoted by $R_{\theta,A}$. We can determine the transformation $R_{\theta,A}$ in three steps:

1) Translate the point A(h,k) to the origin O, so that the center of rotation A is at the origin.
2) Perform the required rotation of θ degrees about the origin, and
3) Translate the origin back to the original position A(h,k).

Using **v**=h**I**+k**J** as the translation vector, we have the following sequence of three transformations:

$$R_{\theta,A} = T_{-v}.\, R_\theta.\, T_v$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -h & -k & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ h & k & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ (1-\cos\theta).h+k.\sin\theta & (1-\cos\theta).k-h.\sin\theta & 1 \end{pmatrix} \quad \text{-------(23)}$$

Example 5: Perform a 45^0 rotation of a triangle A (0,0), B (1,1), C (5,2) about an arbitrary point P(–1, –1).

Solution: Given triangle ABC, as show in Figure (a), can be represented in homogeneous coordinates of vertices as:

$$[ABC] = \begin{array}{c} A \\ B \\ C \end{array}\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 5 & 2 & 1 \end{pmatrix}$$

From equation (23), a rotation matrix R_Q, A about a given arbitrary point A (h, k) is:

$$R_q, A = \begin{pmatrix} Cos\,\theta & Sin\theta & 0 \\ -Sin\,\theta & Cos\,\theta & 0 \\ (1-Cos\theta).h + k.Sin\theta & (1-Cos\theta).k - h.Sin\theta & 1 \end{pmatrix}$$

Thus $R_{45^o,}\ A = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -1 & (\sqrt{2}-1) & 1 \end{pmatrix}$

So the new coordinates $[A'B'C']$ of the rotated triangle [ABC] can be found as:

$$[A'B'C'] = [ABC]\ .\ R_{45^o,A} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 5 & 2 & 1 \end{pmatrix} . \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -1 & (\sqrt{2}-1) & 1 \end{pmatrix} =$$

$$\begin{array}{c} A' \\ B' \\ C' \end{array}\begin{bmatrix} -1 & (\sqrt{2}-1) & 1 \\ -1 & 2\sqrt{2}-1 & 1 \\ \left(\frac{3}{2}\sqrt{2}-1\right) & \left(\frac{9}{2}.\sqrt{2}-1\right) & 1 \end{bmatrix}$$

Thus, $A' = (-1,\ \sqrt{2}-1)$, $B' = (-1,\ 2\sqrt{2}-1)$, and $C' = \left(\frac{3}{2}\sqrt{2}-1, \frac{9}{2}\sqrt{2}-1\right)$. The following *figure (a) and (b)* shows a given triangle, before and after the rotation.

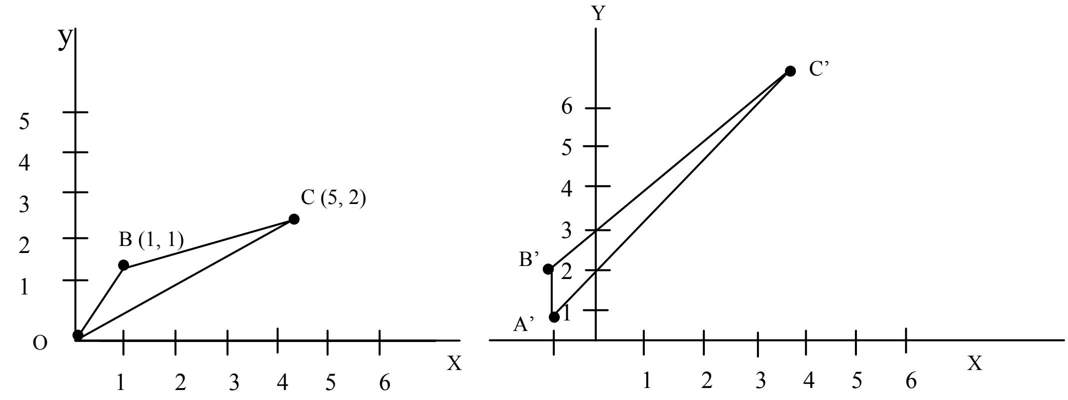


Figure (a) **Figure (b)**

1.3.2 Reflection about a Line

Reflection is a transformation which generates the mirror image of an object. As discussed in the previous block, the mirror reflection helps in achieving 8-way symmetry for the circle to simplify the scan conversion process. For reflection we need to know the reference axis or reference plane depending on whether the object is 2-D or 3-D.

Let the line L be represented by y=mx+c, where 'm' is the slope with respect to the x axis, and 'c' is the intercept on y-axis, as shown in *Figure 7*. Let P'(x',y') be the mirror reflection about the line L of point P(x,y).

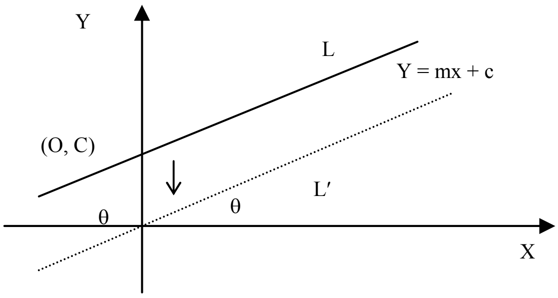


Figure 7

The transformation about mirror reflection about this line L consists of the following basic transformations:

1) Translate the intersection point A(0,c) to the origin, this shifts the line L to L'.
2) Rotate the shifted line L' by –θ degrees so that the line L' aligns with the x-axis.
3) Mirror reflection about x-axis.
4) Rotate the x-axis back by θ degrees
5) Translate the origin back to the intercept point (0,c).

In transformation notation, we have

$$M_L = T_{-v}.R_{-\theta}.M_X.R_{\theta}.T_v \quad , \quad \text{where } v=0I+cJ$$

$$M_L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta - \sin^2\theta & 2.\cos\theta.\sin\theta & 0 \\ 2.\sin\theta.\cos\theta & \sin^2\theta - \cos^2\theta & 0 \\ -2.c.\sin\theta.\cos\theta & -c.(\sin^2\theta - \cos^2\theta)+c & 1 \end{pmatrix} \qquad \text{----------(24)}$$

Let tanθ=m, the standard trigonometry yields $\sin\theta = m/\sqrt{(m^2+1)}$ and $\cos\theta = 1/\sqrt{(m^2+1)}$. Substituting these values for sinθ and cosθ in the equation (24), we have:

$$M_L = \begin{pmatrix} (1-m^2)/(m^2+1) & 2m/(m^2+1) & 0 \\ 2m/(m^2+1) & (m^2-1)/(m^2+1) & 0 \\ -2cm/(m^2+1) & 2c/(m^2+1) & 1 \end{pmatrix} \qquad \text{------------(25)}$$

Special cases

1) If we put c = 0 and m=tanθ=0 in the equation (25) then we have the reflection about the line y = 0 i.e. about x-axis. In matrix form:

$$Mx = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{----------(26)}$$

2) If c = 0 and m=tanθ=∞ then we have the reflection about the line x=0 i.e. about y-axis. In matrix form:

$$My = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{---------(27)}$$

4) To get the mirror reflection about the line y = x, we have to put m=1 and c=0. In matrix form:

$$M_{y=x} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{-----------(28)}$$

5) Similarly, to get the mirror reflection about the line y = – x, we have to put m = –1 and c = 0. In matrix form:

$$M_{y=-x} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{-----------(29)}$$

6) The mirror reflection about the Origin (i.e., an axis perpendicular to the xy plane and passing through the origin).

$$M_{org} = \begin{matrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{matrix} \qquad \text{----------(30)}$$

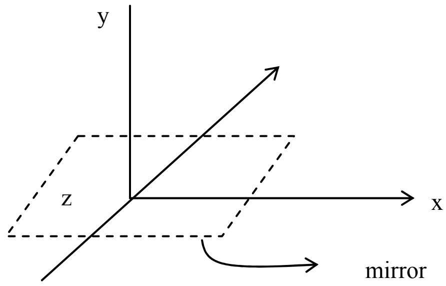


Figure 7(a)

Example 6: Show that two successive reflections about either of the coordinate axes is equivalent to a single rotation about the coordinate origin.

Solution: Let (x, y) be any object point, as shown in *Figure (a).* Two successive reflection of P, either of the coordinate axes, i.e., Reflection about x-axis followed by reflection about y-axis or *vice-versa* can be reprosecuted as:

$$(x, y) \xrightarrow[M_x]{} (x, -y) \xrightarrow[M_y]{} (-x, -y) \qquad \text{----(i)}$$

$$(x, y) \xrightarrow[M_y]{} (x, -y) \xrightarrow[M_x]{} (-x, -y) \qquad \text{----(ii)}$$

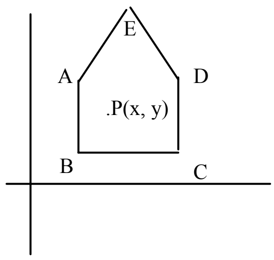
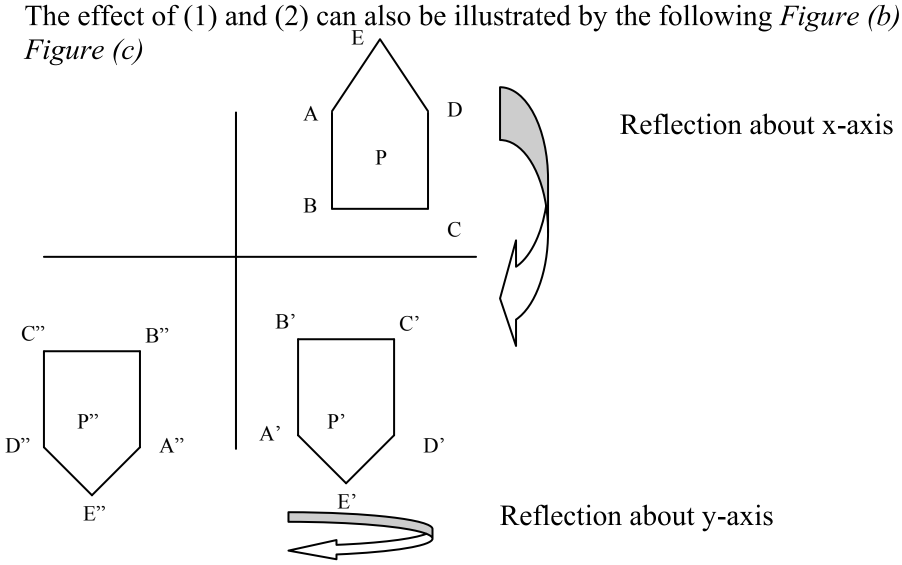


Figure (a)

The effect of (1) and (2) can also be illustrated by the following *Figure (b) and Figure (c)*



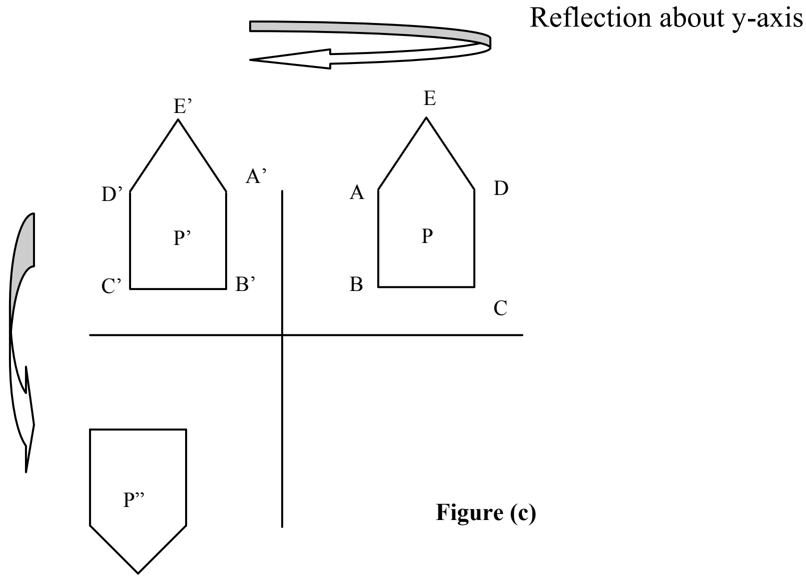


Figure (c)

Reflection about x-axis

From equation (i) and (ii), we can write:

$$(x, y) \longrightarrow (-x, -y) \quad = (x, y)\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{(iii)}$$

Equation (3) is the required reflection about the origin. Hence, two successive reflections about either of the coordinate axes is just equivalent to a single rotation about the coordinate origin.

Example 7: Find the transforation matrix for the reflection about the line y = x.

Solution: The transformation for mirror reflection about the line y = x, consists of the following three basic transformations.

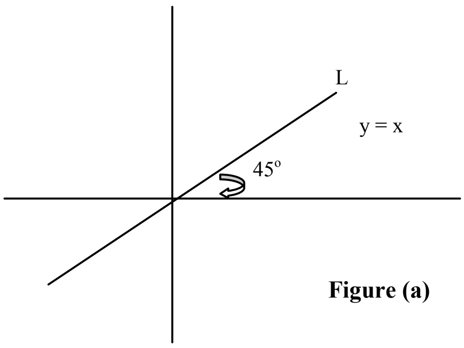


Figure (a)

1) Rotate the line L through 45° in clockwise rotation,
2) Perform the required Reflection about the x-axis.
3) Rotate back the line L by – 45°

i.e.,

$M_L = R_{45^o} \,.\, M_x \,.\, R_{-45^o}$

$$= \begin{bmatrix} \text{Cos}\,45^o & -\text{Sin}\,45^o & 0 \\ \text{Sin}\,45^o & \text{Cos}\,45^o & 0 \\ 0 & 0 & 1 \end{bmatrix} \,.\, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \,.\, \begin{bmatrix} Cos\,45^o & +Sin\,45^o & 0 \\ -Sin\,45^o & Cos\,45^o & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \text{Cos}\,45^o & \text{Sin}\,45^o & 0 \\ \text{Sin}\,45^o & -\text{Cos}\,45^o & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} \text{Cos}\,45^o & \text{Sin}\,45^o & 0 \\ -\text{Sin}\,45^o & \text{Cos}\,45^o & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \text{Cos}\,90 & \text{Sin}\,90 & 0 \\ \text{Sin}\,90 & -\text{Cos}\,90 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = M_y = x$$

Example 8 : Reflect the diamond-shaped polygon whose vertices are A(–1,0), B(0, –2),C(1,0) and D(0,2) about (a) the horizontal line y=2, (b) the vertical line x=2, and (c) the line y=x+2.

Solution: We can represent the given polygon by the homogeneous coordinate matrix as

$$V=[ABCD] = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

a) The horizontal line y=2 has an intercept (0,2) on y axis and makes an angle of 0 degree with the x axis. So m=0 and c=2. Thus, the reflection matrix

$M_L = T_{-v}.R_{-\theta}.M_x.R_{\theta}.T_{-v}$, where $\mathbf{v}=0\mathbf{I}+2\mathbf{J}$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 4 & 1 \end{pmatrix}$$

So the new coordinates A’B’C’D’ of the reflected polygon ABCD can be found as:

[A’B’C’D’]=[ABCD]. M_L

$$= \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 4 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 4 & 1 \\ 0 & 6 & 1 \\ 1 & 4 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

Thus, A’=(-1,4), B’=(0,6), C’=(1,4) and D’=(0,2).

b) The vertical line x=2 has no intercept on y-axis and makes an angle of 90 degree with the x-axis. So m=tan90^0=∞ and c=0. Thus, the reflection matrix

$M_L = T_{-v}.R_{-\theta}.M_y.R_{\theta}.T_{-v}$, where $\mathbf{v}=2\mathbf{I}$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}$$

So the new coordinates A’B’C’D’ of the reflected polygon ABCD can be found as:

[A’B’C’D’]=[ABCD]. M_L

$$= \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} . \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 1 \\ 4 & -2 & 1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{pmatrix}$$

Thus, A’=(5,0), B’=(4,-2), C’=(3,0) and D’=(4,2)

c) The line y=x+2 has an intercept (0,2) on y-axis and makes an angle of 45^0 with the x-axis. So $m=\tan 45^0=1$ and c=2. Thus, the reflection matrix

$$M_L=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -2 & 2 & 1 \end{pmatrix}$$

The required coordinates A',B', C', and D' can be found as:
$[A'B'C'D']=[ABCD].\ M_L$

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -2 & 2 & 1 \end{pmatrix}. \quad = \begin{pmatrix} -2 & 1 & 1 \\ -4 & 2 & 1 \\ -2 & 3 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

Thus, A'=(–2,1), B'=(–4,2), C'=(–2,3) and D'=(0,2)

The effect of the reflected polygon, which is shown in *Figure (a)*, about the line y=2, x=2, and y=x+2 is shown in *Figure (b) - (d)*, respectively.

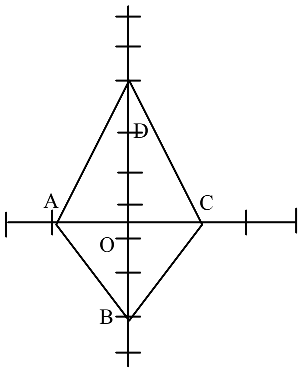


Figure (a)

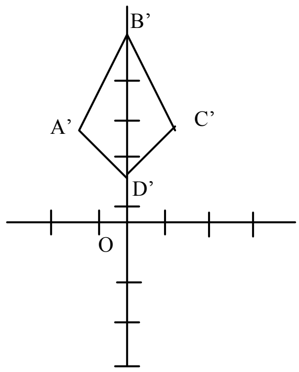


Figure (b)

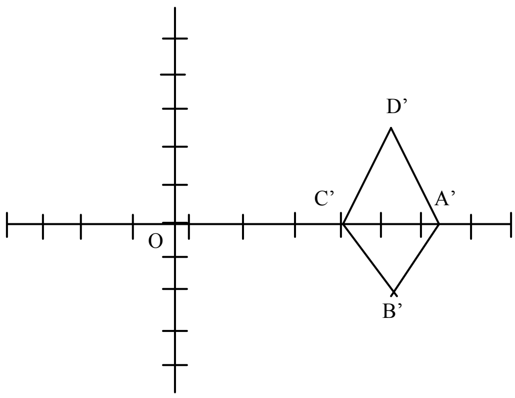


Figure (c)

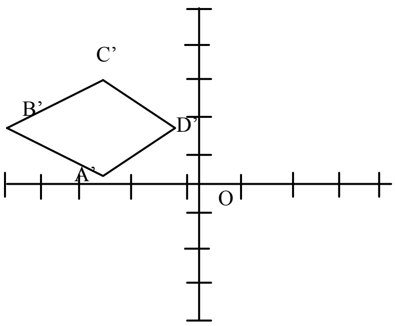


Figure (d)

1.4 HOMOGENEOUS COORDINATE SYSTEMS

Let P(x,y) be any point in 2-D Euclidean (Cartesian) system.

In Homogeneous Coordinate system, we add a third coordinate to a point. Instead of (x,y), each point is represented by a triple (x,y,H) such that H≠0; with the condition that (x1,y1,H1)=(x2,y2,H2) ↔ x1/H1 = x2/H2 ; y1/H1 = y2/H2.

(Here, if we take H=0, then we have point at infinity, i.e., generation of horizons).

Thus, (2,3,6) and (4,6,12) are the same points are represented by different coordinate triples, i.e., each point has many different Homogeneous Coordinate representation.

2-D Euclidian System	Homogeneous Coordinate System
Any point (x,y) ⟶	(x,y,1)
	If (x,y,H) be any point in HCS(such that H≠0); Then (x,y,H)=(x/H,y/H,1)
(x/H,y/H) ⟵	(x,y,H)

Now, we are in the position to construct the matrix form for the translation with the use of homogeneous coordinates.

For translation transformation (x,y)→(x+tx,y+ty) in Euclidian system, where tx and ty are the translation factor in x and y direction, respectively. Unfortunately, this way of describing translation does not use a matrix, so it cannot be combined with other transformations by simple matrix multiplication. Such a combination would be desirable; for example, we have seen that rotation about an arbitrary point can be done by a translation, a rotation, and another translation. We would like to be able to combine these three transformations into a single transformation for the sake of efficiency and elegance. One way of doing this is to use homogeneous coordinates. In homogeneous coordinates we use 3x3 matrices instead of 2x2, introducing an additional dummy coordinate H. Instead of (x,y), each point is represented by a triple (x,y,H) such that H≠0; In two dimensions the value of H is usually kept at 1 for simplicity.

Thus, in HCS (x,y,1) → (x+tx,y+ty,1), now, we can express this in matrix form as:

$$(x',y',1)=(x,y,1)\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ t_x & t_y & 1\end{pmatrix}$$

The ***advantage*** of introducing the matrix form of translation is that it simplifies the operations on complex objects, i.e., we can now build complex transformations by multiplying the basic matrix transformations.

In other words, we can say, that a sequence of transformation matrices can be concatenated into a single matrix. This is an effective procedure as it reduces the computation because instead of applying initial coordinate position of an object to each transformation matrix, we can obtain the final transformed position of an object by applying composite matrix to the initial coordinate position of an object. Matrix representation is standard method of implementing transformations in computer graphics.

Thus, from the point of view of matrix multiplication, with the matrix of translation, the other basic transformations such as scaling, rotation, reflection, etc., can also be expressed as 3x3 *homogeneous coordinate matrices*. This can be accomplished by augmenting the 2x2 matrices with a third row (0,0,x) and a third column. That is

$$\begin{pmatrix}0\\0\\1\end{pmatrix} \qquad\qquad \begin{pmatrix}a & b & 0\\ c & d & 0\\ 0 & 0 & 1\end{pmatrix}$$

Example 9: Show that the order in which transformations are performed is important by applying the transformation of the triangle ABC by:

(i) Rotating by 45^o about the origin and then translating in the direction of the vector (1,0), and

(ii) Translating first in the direction of the vector (1,0), and then rotating by 45^o about the origin, where A = (1, 0) B = (0 ,1) and C = (1, 1).

Solution: We can represent the given triangle, as shown in *Figure (a)*, in terms of Homogeneous coordinates as:

$$[ABC] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

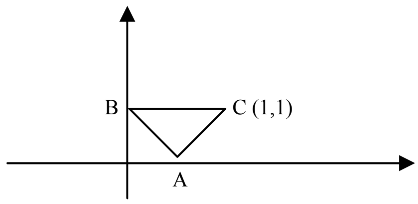


Figure (a)

Suppose the rotation is made in the counter clockwise direction. Then, the transformation matrix for rotation, R_{45^o}, in terms of homogeneous coordinate system is given by:

$$R_{45^o} = \begin{bmatrix} \text{Cos}\,45^o & \text{Sin}\,45^o & 0 \\ -\text{Sin}\,45^o & \text{Cos}\,45^o & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the Translation matrix, T_v, where V = 1I + 0J is:

$$T_v = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

where t_x and t_y is the translation factors in the x and y directions respectively.

i) Now the rotation followed by translation can be computed as:

$$R_{45^o} \,.\, T_v = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

So the new coordinates $A'B'C'$ of a given triangle ABC can be found as:

$$[A'B'C'] = [ABC] \,.\, R_{45^o} \,.\, T_v$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} . \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} (1/\sqrt{2}+1) & 1/\sqrt{2} & 1 \\ (-1/\sqrt{2}+1) & 1/\sqrt{2} & 1 \\ 1 & \sqrt{2} & 1 \end{bmatrix} \qquad \text{(I)}$$

implies that the given triangle A(1,0), B (0, 1) C (1, 1) be transformed into $A'\left(\frac{1}{\sqrt{2}}+1,\ \frac{1}{\sqrt{2}}\right)$, $B'\left(\frac{-1}{\sqrt{2}}+1,\ \frac{1}{\sqrt{2}}\right)$ and $C'\left(1,\ \sqrt{2}\right)$, respectively, as shown in *Figure (b)*.

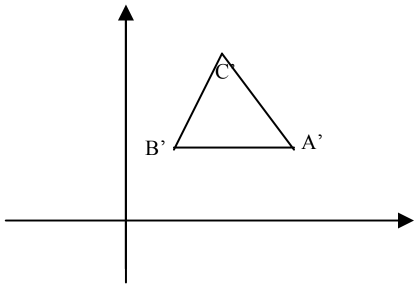


Figure (b)

Similarly, we can obtain the translation followed by rotation transformation as:

$$T_v \,.\, R_{45^o} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1 \end{bmatrix}$$

And hence, the new coordinates $A'B'C'$ can be computed as:

$$[A'B'C'] = [A\,B\,C].T_v \; R_{45^o}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} . \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} & 2/\sqrt{2} & 1 \\ 0 & 2/\sqrt{2} & 1 \\ 1/\sqrt{2} & 3/\sqrt{2} & 1 \end{bmatrix} \qquad \text{(II)}$$

Thus, in this case, the given triangle A(1,0), B(0, 1) and C(1,1) are transformed into $A''\left(2/\sqrt{2}, 2/\sqrt{2}\right)$, $B''\left(0, 2/\sqrt{2}\right)$ *and* $C''\left(\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$, respectively, as shown in *Figure (c)*.

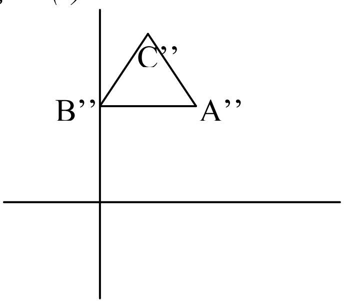


Figure (c)

By (I) and (II), we see that the two transformations do not commute.

☞ Check Your Progress 3

1) Show that transformation matrix (28), for the reflection about the line y=x, is equivalent to the reflection relative to the x-axis followed by a counterclockwise rotation of 90^0.

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2) Give a single 3x3 homogeneous coordinate transformation matrix, which will have the same effect as each of the following transformation sequences.
   a) Scale the image to be twice as large and then translate it 1 unit to the left.
   b) Scale the x direction to be one-half as large and then rotate counterclockwise by 90^0 about the origin.
   c) Rotate counterclockwise about the origin by 90^0 and then scale the x direction to be one-half as large.
   d) Translate down ½ unit, right ½ unit, and then rotate counterclockwise by 45^0.

3) Obtain the transformation matrix for mirror reflection with respect to the line y=ax, where 'a' is a constant.

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4) Obtain the mirror reflection of the triangle formed by the vertices A(0,3),B(2,0) and C(3,2) about the line passing through the points (1,3) and (–1, –1).

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1.5 3-D TRANSFORMATIONS

The ability to represent or display a three-dimensional object is fundamental to the understanding of the shape of that object. Furthermore, the ability to rotate, translate, and project views of that object is also, in many cases, fundamental to the understanding of its shape. Manipulation, viewing, and construction of three-dimensional graphic images require the use of three-dimensional *geometric* and *coordinate transformations*. In *geometric transformation*, the coordinate system is fixed, and the desired transformation of the object is done with respect to the coordinate system. In *coordinate* transformation, the object is fixed and the desired transformation of the object is done on the coordinate system itself. These transformations are formed by composing the basic transformations of translation, scaling, and rotation. Each of these transformations can be represented as a matrix transformation. This permits more complex transformations to be built up by use of matrix multiplication or concatenation. We can construct the complex objects/pictures, by instant transformations. In order to represent all these transformations, we need to use homogeneous coordinates.

Hence, if P(x,y,z) be any point in 3-D space, then in HCS, we add a fourth-coordinate to a point. That is instead of (x,y,z), each point can be represented by a Quadruple (x,y,z,H) such that H≠0; with the condition that x1/H1=x2/H2; y1/H1=y2/H2; z1/H1=z2/H2. For two points $(x_1, y_1, z_1, H_1) = (x_2, y_2, z_2, H_2)$ where $H_1 \neq 0, H_2 \neq 0$. Thus any point (x,y,z) in Cartesian system can be represented by a four-dimensional vector as (x,y,z,1) in HCS. Similarly, if (x,y,z,H) be any point in HCS then (x/H,y/H,z/H) be the corresponding point in Cartesian system. Thus, a point in three-dimensional space (x,y,z) can be represented by a four-dimensional point as: (x',y',z',1)=(x,y,z,1).[T], where [T] is some transformation matrix and (x',y'z',1) is a new coordinate of a given point (x,y,z,1), after the transformation.

The generalized 4x4 transformation matrix for three-dimensional homogeneous coordinates is:

$$[T]=\begin{pmatrix} a & b & c & w \\ d & e & f & x \\ g & h & I & y \\ l & m & n & z \end{pmatrix} = \left(\begin{array}{c|c} (3x3) & (3x1) \\ \hline (1x3) & (1x1) \end{array}\right) \qquad \text{--------(31)}$$

The upper left (3x3) sub matrix produces *scaling, shearing, rotation* and *reflection* transformation. The lower left (1x3) sub matrix produces *translation*, and the upper right (3x1) sub matrix produces a *perspective* transformation, which we will study in the next unit. The final lower right-hand (1x1) sub matrix produces overall *scaling*.

1.5.1 Transformation for 3-D Translation

Let P be the point object with the coordinate (x,y,z). We wish to translate this object point to the new position say, P'(x',y',z') by the translation Vector V=t_x.**I**+t_y.**J**+t_z.**K** , where t_x , t_y and t_z are the translation factor in the x, y, and z directions respectively, as shown in *Figure 8*. That is, a point (x,y,z) is moved to (x+ t_x,y+ t_y,z+ t_z). Thus the new coordinates of a point can be written as:

$$\left.\begin{array}{l} x'=x+t_x \\ y'=y+t_y \\ z'=z+t_z \end{array}\right\} =T_v \qquad \text{---------(32)}$$

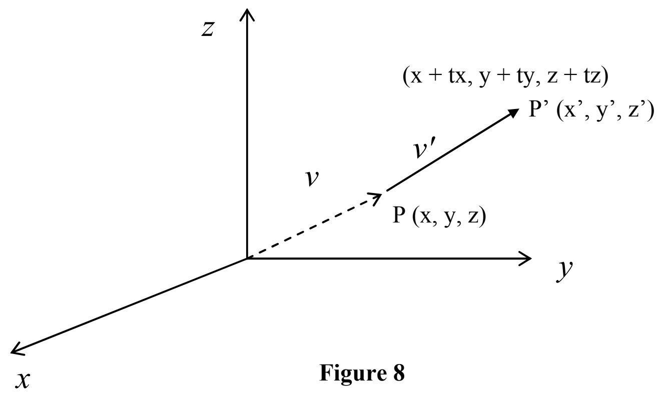


Figure 8

In terms of homogeneous coordinates, equation (32) can be written as

$$(x',y',z',1)=(x,y,z,1)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ t_x & t_y & t_z & 1 \end{pmatrix} \qquad \text{--------(33)}$$

i.e., $P'_h = P_h.T_v$ ----------(34)

1.5.2 Transformation for 3-D Rotation

Rotation in three dimensions is considerably more complex then rotation in two dimensions. In 2-D, a rotation is prescribed by an angle of rotation θ and a centre of rotation, say P.

However, in 3-D rotations, we need to mention the angle of rotation and the axis of rotation. Since, we have now three axes, so the rotation can take place about any one of these axes. Thus, we have rotation about x-axis, y-axis, and z-axis respectively.

Rotation about z-axis

Rotation about z-axis is defined by the xy-plane. Let a 3-D point P(x,y,z) be rotated to P'(x',y',z') with angle of rotation θ see *Figure 9*. Since both P and P' lies on xy-plane i.e., z=0 plane their z components remains the same, that is z=z'=0.

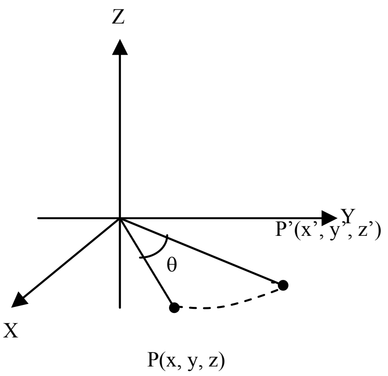


Figure 9

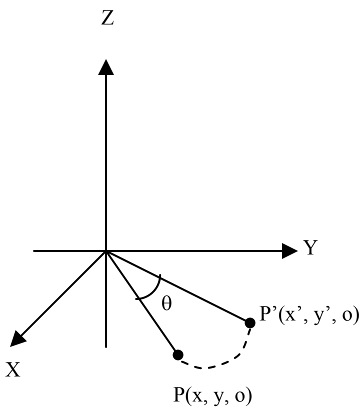


Figure 10

Thus, P'(x'y',0) be the result of rotation of point P(x,y,0) making a positive (anticlockwise) angle ϕ with respect to z=0 plane, as shown in *Figure 10*.

From *figure (10)*,

$P(x,y,0) = P(r.\cos\phi, r.\sin\phi, 0)$
$P'(x',y',0)=P[r.\cos(\phi+\theta), r\sin(\phi+\theta), 0]$

The coordinates of P' are:

$$\begin{aligned} x' &= r.\cos(\theta+\phi)=r(\cos\theta\cos\phi - \sin\theta\sin\phi) \\ &= x.\cos\theta - y.\sin\theta \quad \text{(where } x=r\cos\phi \text{ and } y=r\sin\phi) \end{aligned}$$

similarly;

$$\begin{aligned} y' &= r\sin(\theta+\phi)=r(\sin\theta\cos\phi + \cos\theta.\sin\phi) \\ &= x\sin\theta+y\cos\theta \end{aligned}$$

Thus,

$$[Rz]_\theta = \left\{ \begin{array}{l} x' = x.\cos\theta - y.\sin\theta \\ y' = x\sin\theta + y\cos\theta \\ z'=z \end{array} \right\} \qquad \text{----------(35)}$$

In matrix form,

$$(x'y',z')=(x,y,z) \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{----------(36)}$$

In terms of HCS, equation (36) becomes

$$(x'y',z',1)=(x,y,z,1) \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{----------(37)}$$

That is, $P'_h = P_h.[Rz]_\theta$ ---------(38)

Rotations about x-axis and y-axis

Rotation about the x-axis can be obtained by cyclic interchange of x→y→z→x in equation (35) of the z-axis rotation i.e.,

$$[Rz]_\theta = \begin{cases} x' = x.\cos\theta - y.\sin\theta \\ y' = x\sin\theta + y\cos\theta \\ z' = z \end{cases}$$

$\Big\downarrow$ After cyclic interchange of x→y→z→x

$$[Rx]_\theta = \begin{cases} y' = y.\cos\theta - z.\sin\theta \\ z' = y.\sin\theta + z.\cos\theta \\ x' = x \end{cases} \quad \text{--------(39)}$$

So, the corresponding transformation matrix in homogeneous coordinates becomes

$$(x'y',z',1) = (x,y,z,1)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

That is, $P'_h = P_h.[Rx]_\theta$ -------(40)

Similarly, the rotation about y-axis can be obtained by cyclic interchange of x→y→z→x in equation (39) of the x-axis rotation $[Rx]_\theta$ i.e.,

$$[Rx]_\theta = \begin{cases} y' = y.\cos\theta - z.\sin\theta \\ z' = y.\sin\theta + z.\cos\theta \\ x' = x \end{cases}$$

$\Big\downarrow$ After cyclic interchange of x→y→z→x

$$[R_y]_\theta = \begin{cases} z' = z.\cos\theta - x.\sin\theta \\ x' = z.\sin\theta + x.\cos\theta \\ y' = y \end{cases} \quad \text{-------(41)}$$

So, the corresponding transformation matrix in homogeneous coordinates becomes

$$(x'y',z',1) = (x,y,z,1)\begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

That is, $P' = P.[R_y]_\theta$ -------(42)

1.5.3 Transformation for 3-D Scaling

As we have seen earlier, the scaling process is mainly used to change the size of an object. The scale factors determine whether the scaling is a magnification, s>1, or a reduction, s<1. Two-dimensional scaling, as in equation (8), can be easily extended to scaling in 3-D case by including the z-dimension.

For any point (x,y,z), we move into $(x.s_x, y.s_y, z.s_z)$, where s_x, s_y, and s_z are the scaling factors in the x,y, and z-directions respectively.

Thus, scaling with respect to origin is given by:

$$S_{s_x,s_y,s_z} = \begin{cases} x' = x.s_x \\ y' = y.s_y \\ z' = z.s_z \end{cases} \qquad \text{---------(43)}$$

In matrix form,

$$(x'y',z') = (x,y,z) \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{pmatrix} \qquad \text{---------(44)}$$

In terms of HCS, equation (44) becomes

$$(x'y',z',1) = (x,y,z,1) \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

That is, $P' = P.\, S_{s_x,s_y,s_z}$ --------(45)

1.5.4 Transformation for 3-D Shearing

Two-dimensional xy- shearing transformation, as defined in equation (19), can also be easily extended to 3-D case. Each coordinate is translated as a function of displacements of the other two coordinates. That is,

$$Sh_{xyz} = \begin{cases} x' = x + a.y + b.z \\ y' = y + c.x + d.z \\ z' = z + e.x + f.y \end{cases} \qquad \text{--------(46)}$$

where a,b,c,d,e and f are the shearing factors in the respective directions.

In terms of HCS, equation (46) becomes

$$(x'y',z',1) = (x,y,z,1) \begin{pmatrix} 1 & c & e & 0 \\ a & 1 & f & 0 \\ b & d & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

That is, $P'_h = P_h.Sh_{xyz}$ -------(47)

Note that the off-diagonal terms in the upper left 3x3 sub matrix of the generalized 4x4 transformation matrix in equation (31) produce shear in three dimensions.

1.5.5 Transformation for 3-D Reflection

For 3-D reflections, we need to know the reference plane, i.e., a plane about which the reflection is to be taken. Note that for each reference plane, the points lying on the plane will remain the same after the reflection.

Mirror reflection about xy-plane

Let P(x,y,z) be the object point, whose mirror reflection is to be obtained about xy-plane(or z=0 plane). For the mirror reflection of P about xy-plane, only there is a change in the sign of z-coordinate, as shown in *Figure (11)*. That is,

$$M_{xy} = \begin{cases} x' = x \\ y' = y \\ z' = -z \end{cases} \qquad \text{-------(48)}$$

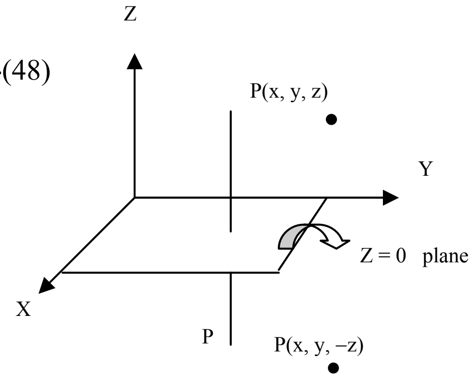


Figure 11

In matrix form,

$$(x'y',z') = (x,y,z)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \text{----(49)}$$

In terms of HCS (Homogenous coordinate systems), equation (49) becomes

$$(x'y',z',1) = (x,y,z,1)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

That is, $P' = P.M_{xy}$ -----(50)

Similarly, the mirror reflection about yz plane shown in *Figure 12* can be represented as:

$$M_{yz} = \begin{cases} x' = -x \\ y' = y \\ z' = z \end{cases} \qquad \text{-----(51)}$$

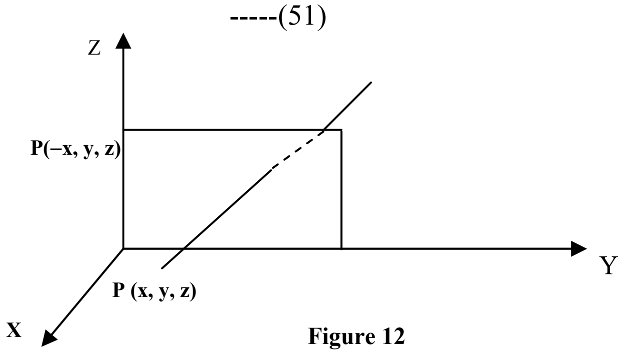


Figure 12

In matrix form,

$$(x'y',z') = (x,y,z)\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{------(52)}$$

In terms of HCS, equation (52) becomes

$$(x'y',z',1) = (x,y,z,1)\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

That is, $P' = P.\ M_{yz}$; ------(53)

and similarly, the reflection about xz plane, shown in *Figure 13*, can be presented as:

$$M_{xz} = \begin{cases} x' = x \\ y' = -y \\ z' = z \end{cases} \quad \text{-------(54)}$$

In matrix form,

$$(x'y',z') = (x,y,z)\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{-------(55)}$$

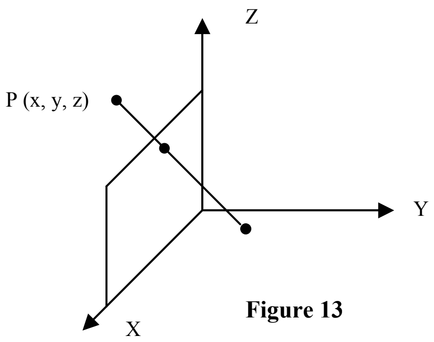


Figure 13

In terms of HCS, equation (55) becomes

$$(x'y',z',1) = (x,y,z,1)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

That is, $P' = P.\ M_{xz}$ --------(56)

1.6 SUMMARY

In this unit, the following things have been discussed in detail:

- Various geometric transformations such as translation, rotation, reflection, scaling and shearing.
- Translation, Rotation and Reflection transformations are used to manipulate the given object, whereas Scaling and Shearing transformation changes their sizes.
- Translation is the process of changing the position (not the shape/size) of an object w.r.t. the origin of the coordinate axes.
- In 2-D rotation, an object is rotated by an angle θ. There are two cases of 2-D rotation: *case1*- rotation about the origin and *case2*- rotation about an arbitrary point. So, in 2-D, a rotation is prescribed by an angle of rotation θ and a centre of rotation, say P. However, in 3-D rotations, we need to mention the angle of rotation and the axis of rotation.
- Scaling process is mainly used to change the shape/size of an object. The scale factors determine whether the scaling is a magnification, $s>1$, or a reduction, $s<1$.
- Shearing transformation is a special case of translation. The effect of this transformation looks like "pushing" a geometric object in a direction that is parallel to a coordinate plane (3D) or a coordinate axis (2D). How far a direction is pushed is determined by its *shearing factor.*
- Reflection is a transformation which generates the mirror image of an object. For reflection we need to know the reference axis or reference plane depending on whether the object is 2-D or 3-D.
- Composite transformation involves more than one transformation concatenated into a single matrix. This process is also called *concatenation of matrices.* Any transformation made about an arbitrary point makes use of composite transformation such as Rotation about an arbitrary point, reflection about an arbitrary line, etc.
- The use of homogeneous coordinate system to represent the translation transformation in matrix form, extends our N-coordinate system with (N+1) coordinate system.

- The transformations such as translation, rotation, reflection, scaling and shearing can be extended to 3D cases.

1.7 SOLUTIONS/ANSWERS

Check Your Progress 1

1) Matrix representation are standard method of implementing transformations in computer graphics. But unfortunately, we are not able to represent all the transformations in a (2 x 2) matrix form; such as translation. By using Homogeneous coordinates system (HCS), we can represent all the transformations in matrix form. For translation of point (x, y) $\rightarrow$ (x +t_x, y + t_y), it is not possible to represent this transformation in matrix form. But, now in HCS;

$$(x',\ y',\ 1) = (x, y, 1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix}$$

The advantage of introducing the matrix form for translation is that we can now build a complex transformation by multiplying the basic matrix transformation. This is an effective procedure as it reduces the computations.

2) The translation factor, t_x and t_y can be obtained from new old coordinates of vertex C.

$t_x = 6 \ -1 \ = \ 5$

$t_y = 7 \ -1 \ = 6$

The new coordinates [A′ B′ C′ D′] = [A B C D] . T_v

$$\begin{array}{c} A' \\ B' \\ C' \\ D' \end{array}\begin{bmatrix} x'_1 & y'_1 & 1 \\ x'_2 & y'_2 & 1 \\ x'_3 & y'_3 & 1 \\ x'_4 & y'_4 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 1 \\ 5 & 7 & 1 \\ 6 & 7 & 1 \\ 6 & 6 & 1 \end{bmatrix}$$

Thus A′ = (5, 6), B′ = (5, 7), C′ = (6, 7) and D′ = (6, 6)

3) The new coordinate P′ of a point P, after the Rotation of 45° is:

$P' = P.R_{45^o}$

$$(x',\ y', 1) = (x,\ y, 1)\begin{bmatrix} \text{Cos}45^o & \text{Sin}45^o & 0 \\ -\text{Sin}45^o & \text{Cos}45^o & 0 \\ 0 & 0 & 1 \end{bmatrix} = (x, y, 1)\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \left[\frac{1}{\sqrt{2}}(x - y),\ \frac{1}{\sqrt{2}}(x + y,\ 1)\right] = \left(0,\ 6/\sqrt{2},\ 1\right)$$

Now, this point P′is again translated by t_x = 5 and t_y = 6. So the final coordinate P″ of a given point P, can be obtained as:

$$(x'',\ y'',\ 1) = (x',\ y',\ 1)\ .\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 6 & 1 \end{bmatrix}$$

$$= \left(0,\ 6/\sqrt{2},1\right).\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\5 & 6 & 1\end{bmatrix} = \left(5,\frac{6}{\sqrt{6}}+6,\ 1\right)$$

Thus $P''(x'', y'') = (5,\frac{6}{\sqrt{2}}+6)$

4) $R_\theta = \begin{pmatrix}\text{Cos}\theta & \text{Sin}\theta\\-\text{Sin}\theta & \text{Cos}\theta\end{pmatrix}$ $R_{-\theta} = \begin{pmatrix}\text{Cos}(-\theta) & \text{Sin}(-\theta)\\-\text{Sin}(-\theta) & \text{Cos}(\theta)\end{pmatrix} = \begin{pmatrix}\text{Cos}\theta & -\text{Sin}\theta\\\text{Sin}\theta & \text{Cos}\theta\end{pmatrix}$

$$\because\quad R_\theta . R_{-\theta} = \begin{pmatrix}\text{Cos}\theta & \text{Sin}\theta\\-\text{Sin}\theta & \text{Cos}\theta\end{pmatrix}\begin{pmatrix}\text{Cos}\theta & -\text{Sin}\theta\\-\text{Sin}\theta & \text{Cos}\theta\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \text{Identity matrix}$$

Therefore, we can say that $R_\theta . R_{-\theta}$ are inverse because $R_\theta . R_{-\theta} = I$. So $R_{-\theta} = R_\theta^{-1}$ i.e., inverse of a rotation by θ degree is a rotation in the opposite direction.

Check Your Progress 2

1) Scaling transformation is mainly used to change the size of an object. The scale factors determines whether the scaling is a compression, S < 1 or a enlargement, S > 1, whereas the effect of shearing is "pushing" a geometric object is a direction parallel to the coordinate axes. Shearing factor determines, how far a direction is pushed.

2) $S_{a,b} = \begin{pmatrix}a & 0\\0 & b\end{pmatrix}$, $S_{c,d} = \begin{pmatrix}c & 0\\0 & d\end{pmatrix}$ and $S_{ac,bd} = \begin{pmatrix}a.c & 0\\0 & b.d\end{pmatrix}$

since

$$S_{a,b}.\ S_{c,d} = \begin{pmatrix}a & 0\\0 & b\end{pmatrix}.\begin{pmatrix}c & 0\\0 & d\end{pmatrix} = \begin{pmatrix}a.c & 0\\0 & b.d\end{pmatrix} \quad -(1)$$

and $S_{c,d}.\ S_{a,b} = \begin{pmatrix}c & 0\\0 & d\end{pmatrix}.\begin{pmatrix}a & 0\\0 & b\end{pmatrix} = \begin{pmatrix}c.a & 0\\0 & d.b\end{pmatrix} \quad -(2)$

from (1) and (2) we can say:

$$S_{a,b}.\ S_{c,d} = S_{c,d}.\ S_{a,b} = S_{ac,\ bd}$$

3)

a) Shift an image to the right by 3 units

$$\therefore S = \begin{pmatrix}0 & 0 & 0\\0 & 1 & 0\\3 & 0 & 1\end{pmatrix}$$

b) Shift the image up by 2 units and down by 1 units i.e. $S_x = S_x + 2$ and $S_y = S_y - 1$

$$\therefore S = \begin{pmatrix} (S_x+2) & 0 & 0 \\ 0 & (S_y-1) & 0 \\ 0 & 0 & 1 \end{pmatrix} \therefore S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

c) Move the image down 2/3 units and left 4 units

$$\therefore S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & -2/3 & 1 \end{pmatrix}$$

4) $S_{Sx,\,Sy} = \begin{pmatrix} S_x & 0 \\ 0 & S_y \end{pmatrix}$ and $R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

we have to find out condition under which $S_{Sx,\,Sy}.\ R_\theta = R_{\theta}.\ S_s,\ S_y$

so $S_{Sx,\,Sy}.\ R_\theta = \begin{pmatrix} S_x & o \\ o & S_y \end{pmatrix} . \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} S_x.\cos\theta & S_x.\sin\theta \\ -S_y.\sin\theta & S_y.\cos\theta \end{pmatrix}$ — (1)

and $R_\theta.\ S_{Sx,Sy} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} . \begin{pmatrix} S_x & o \\ o & S_y \end{pmatrix} = \begin{pmatrix} \cos\theta.S_x & Sin\theta.S_y \\ -S_x.Sin\theta & \cos\theta.S_y \end{pmatrix}$ — (2)

In order to satisfy $S_{Sx,\,Sy.}\ R\theta = R\theta.\ S_{Sx,Sy}$

We have $Sy.\ \sin\theta = \sin\theta.S_x \Rightarrow$ either $\sin\theta = 0$ or $\theta = n\,\pi$, where n is an integer.

$\sin\theta\,(Sy - S_x) = 0$ or $S_x = S_y$ i.e. scaling transform is uniform.

5) No, since $Sh_x(a).\ Sh_y(b) = \begin{pmatrix} 1 & o \\ a & 1 \end{pmatrix} . \begin{pmatrix} 1 & b \\ o & 1 \end{pmatrix} = \begin{pmatrix} 1 & b \\ a & ab+1 \end{pmatrix}$ — (1)

$Sh_y(b).Sh_x(a) = \begin{pmatrix} 1 & b \\ o & 1 \end{pmatrix} \begin{pmatrix} 1 & o \\ a & 1 \end{pmatrix} = \begin{pmatrix} 1+ba & b \\ a & 1 \end{pmatrix}$ — (2)

and $Sh_{xy}(a, b) = \begin{pmatrix} 1 & b \\ a & 1 \end{pmatrix}$

from (1) , (2) and (3), we can say that

$Sh_{xy}(a, b) \neq Sh_x(a)\ .\ Sh_y(b) \neq Sh_y(b).\ Sh_x(a)$

Check Your Progress 3

1) $M_{y=x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ M_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and

Counter clockwise Rotation of 90°; $R_{90^o} = \begin{bmatrix} \text{Cos}\,90^o & \text{Sin}\,90^o \\ -\text{Sin}\,90^o & \text{Cos}\,90^o \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

We have to show that

$M_y = x = M_x \, . \, R_{90^o}$

Since $M_x \, . \, R_{90^o} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} . \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = M_y = x$

Hence, a reflection about the line y = x, is equivalent to a reflection relative to the x-axis followed by a counter clockwise rotation of 90°.

2) The required single (3 x 3) homogeneous transformation matrix can be obtained as follows:

a) $T = S_{2,2} \, . \, T_{tx-1,\, ty} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

b) $T = S_{s_x + \frac{3}{2}, s_y} . R_{90^o} = \begin{bmatrix} 3/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} Cos 90^o & Sin 90^o & 0 \\ -Sin 90^o & Cos 90^o & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3/2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c) $T = R_{90^o} \, . \, S_{S_x + \frac{3}{2}, S_y} = \begin{bmatrix} Cos 90^o & Sin 90^o & 0 \\ -Sin 90^o & Cos 90^o & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 3/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

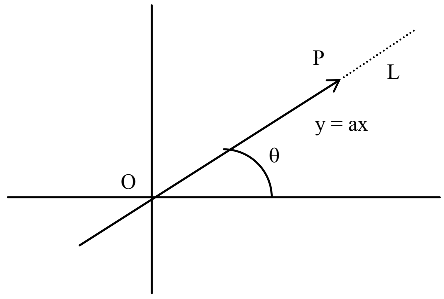
$= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 3/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 & 0 \\ 3/2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$T = T_{tx \frac{-1}{2}, t_y + \frac{1}{2}} \, . \, R_{45^o} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & 1 \end{bmatrix} . \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1\sqrt{2} & 0 & 1 \end{bmatrix}$

3) Let OP be given line L, which makes an angle θ with respect to



The transformation matrix for reflection about an arbitrary line y = mx + c is (see equation 25).

$$M_L = \begin{bmatrix} \frac{1-m^2}{m^2+1} & \frac{2m}{m^2+1} & 0 \\ \frac{2m}{m^2+1} & \frac{m^2-1}{m^2+1} & 0 \\ \frac{-2cm}{m^2+1} & \frac{-2C}{m^2+1} & 1 \end{bmatrix} \quad where\ m = \tan\theta$$

For line y = ax; m = tanθ = a and intercept on y-axis is 0 i.e. c = 0. Thus, transformation matrix for reflection about a line y = ax is:

$$M_L = M_{y=ax} = \begin{bmatrix} \frac{1-a^2}{a^2+1} & \frac{2a}{a^2+1} & 0 \\ \frac{2.a}{a^2+1} & \frac{a^2-1}{a^2+1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{where } a = \tan\theta = m$$

4) The equation of the line passing through the points (1,3) and (–1, –1) is obtained as:

$$y = 2x + 1 \qquad (1)$$

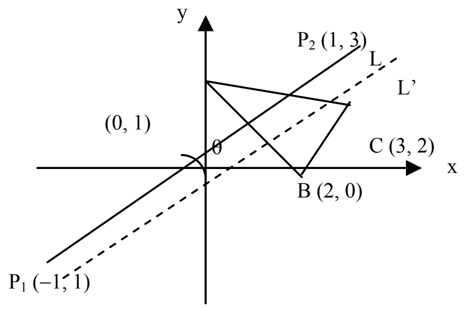


Figure (a)

If θ is the angle made by the line (1) with the positive x-axis, then

$\tan\theta = 2 \Rightarrow \text{Cos}\theta = \frac{1}{\sqrt{2}}$ and $\text{Sin}\theta\ \frac{2}{\sqrt{5}}$

To obtain the reflection about the line (1), the following sequence of transformations can be performed:

1) Translate the intersection point (0, 1) to the origin, this shift the line L to L′
2) Rotate the shifted line L′ by $-\theta^o$ (i.e. clockwise), so that the L′aligns with the x-axis.

3) Perform the reflection about x-axis.
4) Apply the inverse of the transformation of step (2).
5) Apply the inverse of the transformation of step (1).

By performing step 1 – step 5, we get

$$M_L = T_V \,.\, R_\theta \; M_X \,.\, R_\theta^{-1} \,.\, T_V^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} . \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} & 0 \\ 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3/\sqrt{5} & 4/5 & 0 \\ 4/5 & 3/5 & 0 \\ -4/5 & 2/5 & 1 \end{bmatrix}$$

So the new coordinates A′B′C′ of the reflected triangle ABC can be found as:
[A′ B′ C′] = [ABC] . M_L

$$= \begin{bmatrix} 0 & 3 & 1 \\ 2 & 0 & 1 \\ 3 & 2 & 1 \end{bmatrix} . \begin{bmatrix} -3/\sqrt{5} & 4/5 & 0 \\ 4/5 & 3/5 & 0 \\ -4/5 & 2/5 & 1 \end{bmatrix} = \begin{bmatrix} 8/5 & 11/5 & 1 \\ -2 & 2 & 1 \\ -1 & 4 & 1 \end{bmatrix}$$

Thus, $A' = \left(8/5,\ \dfrac{11}{5}\right)$, $B' = (-2, 2)$ and $C' = (-1, 4)$, which is shown in *Figure (b)*.

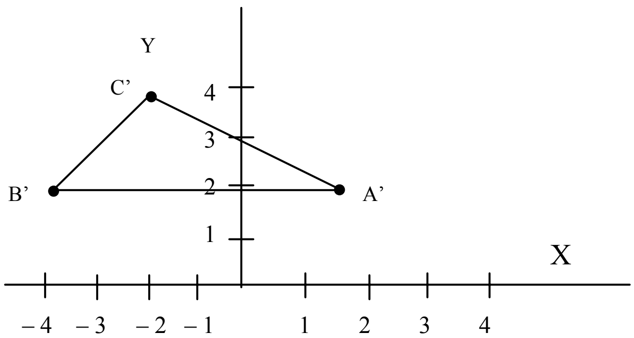


Figure (b)

UNIT 2 VIEWING TRANSFORMATION

Structure **Page Nos.**

2.0 INTRODUCTION

In unit 1, we have discussed the geometric transformations such as Translation, Rotation, Reflection, Scaling and Shearing. Translation, Rotation and Reflection transformations are used to manipulate the given object, whereas Scaling and Shearing transformations are used to modify the shape of an object, either in 2-D or in 3-Dimensional.

A transformation which maps 3-D objects onto 2-D screen, we are going to call it *Projections.* We have two types of Projections namely, *Perspective projection* and *Parallel projection*. This categorisation is based on the fact whether rays coming from the object converge at the centre of projection or not. If, the rays coming from the object converge at the centre of projection, then this projection is known as *Perspective projection*, otherwise it is *Parallel projection*. In the case of parallel projection the rays from an object converge at infinity, unlike perspective projection where the rays from an object converge at a finite distance (called COP).

Parallel projection is further categorised into *Orthographic* and *Oblique projection*. Parallel projection can be categorized according to the angle that the direction of projection makes with the projection plane If the direction of projection of rays is perpendicular to the projection plane then this parallel projection is known as *Orthographic projection* and if the direction of projection of rays is not perpendicular to the projection plane then this parallel projection is known as *Oblique projection*. The orthographic (perpendicular) projection shows only the front face of the given object, which includes only two dimensions: length and width. The oblique projection, on the other hand, shows the front surface and the top surface, which includes three dimensions: length, width, and height. Therefore, an oblique projection is one way to show all three dimensions of an object in a single view.

Isometric projection is the most frequently used type of *axonometric projection*, which is a method used to show an object in all three dimensions (length, width, and height) in a single view. Axonometric projection is a form of orthographic projection in which the projectors are always perpendicular to the plane of projection.

2.1 OBJECTIVES

After going through this unit, you should be able to:

- define the projection;
- categorize various types of Perspective and Parallel projections;
- develop the general transformation matrix for parallel projection;
- describe and develop the transformation for Orthographic and oblique parallel projections;
- develop the transformations for multiview (front, right, top, rear, left and bottom view) projections;
- define the foreshortening factor and categorize the oblique projection on the basis of foreshortening factors;
- derive the transformations for general perspective projection;
- describe and derive the projection matrix for single-point, two-point and three-point perspective transformations, and
- identify the vanishing points.

2.2 PROJECTIONS

Given a 3-D object in a space, Projection can be defined as a mapping of 3-D object onto 2-D viewing screen. Here, 2-D screen is known as Plane of projection or view plane, which constitutes the display surface. The mapping is determined by projection rays called the projectors. Geometric projections of objects are formed by the intersection of lines (called projectors) with a plane called plane of projection /view plane. Projectors are lines from an arbitrary point, called the centre of projection (COP), through each point in an object. *Figure 1* shows a mapping of point P(x,y,z) onto its image P′(x’,y’,z’) in the view plane.

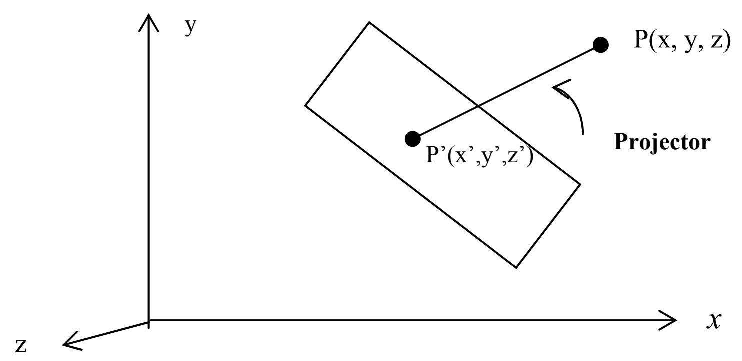


Figure 1

If, the COP (Center of projection) is located at finite point in the three-space, the result is a perspective projection. If the COP is located at infinity, all the projectors are parallel and the result is a parallel projection. *Figure 2(a)-(b)* shows the difference between parallel and perspective projections. In *Figure 2(a)*, **ABCD is projected to A’B’C’D’ on the plane of projection and O is a COP**. In the case of parallel projection the rays from an object converges at infinity, the rays from the object become parallel and will have a direction called ”direction of projection”.

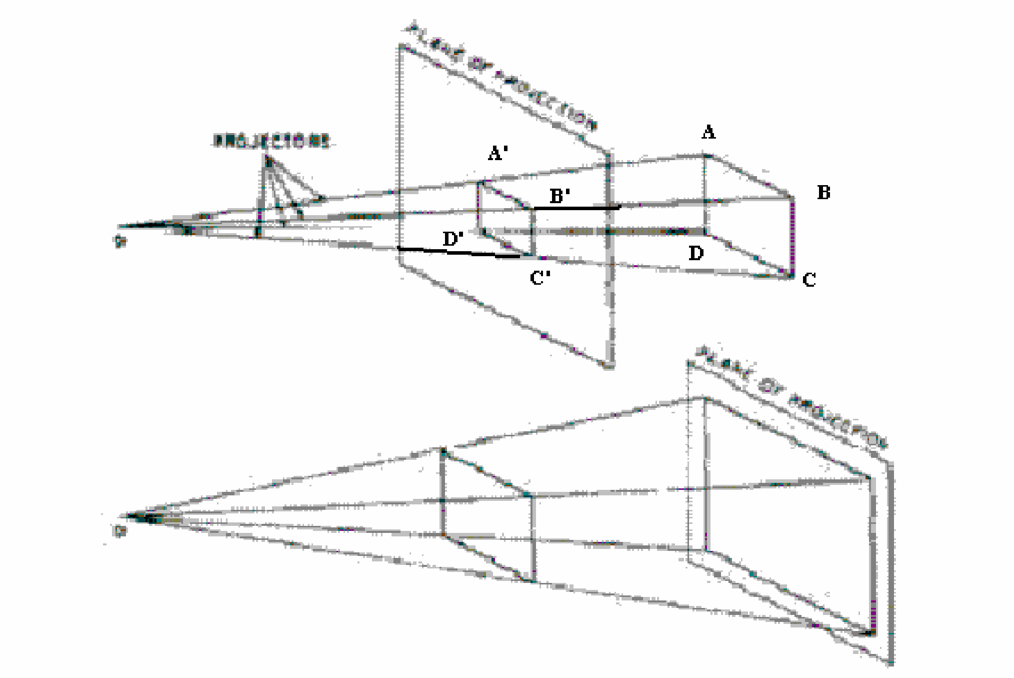


Figure 2(a): Perspective projection

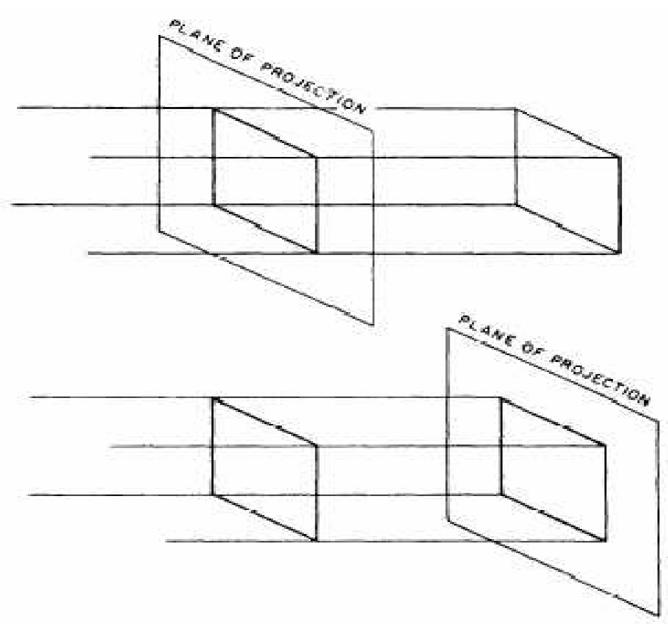


Figure 2(b): Parallel projection

Taxonomy of Projection

There are various types of projections according to the view that is desired. The following *Figure 3* shows taxonomy of the families of *Perspective* and *Parallel* Projections. This categorisation is based on whether the rays from the object converge at COP or not and whether the rays intersect the projection plane perpendicularly or not. The former condition separates the perspective projection from the parallel projection and the latter condition separates the Orthographic projection from the Oblique projection.

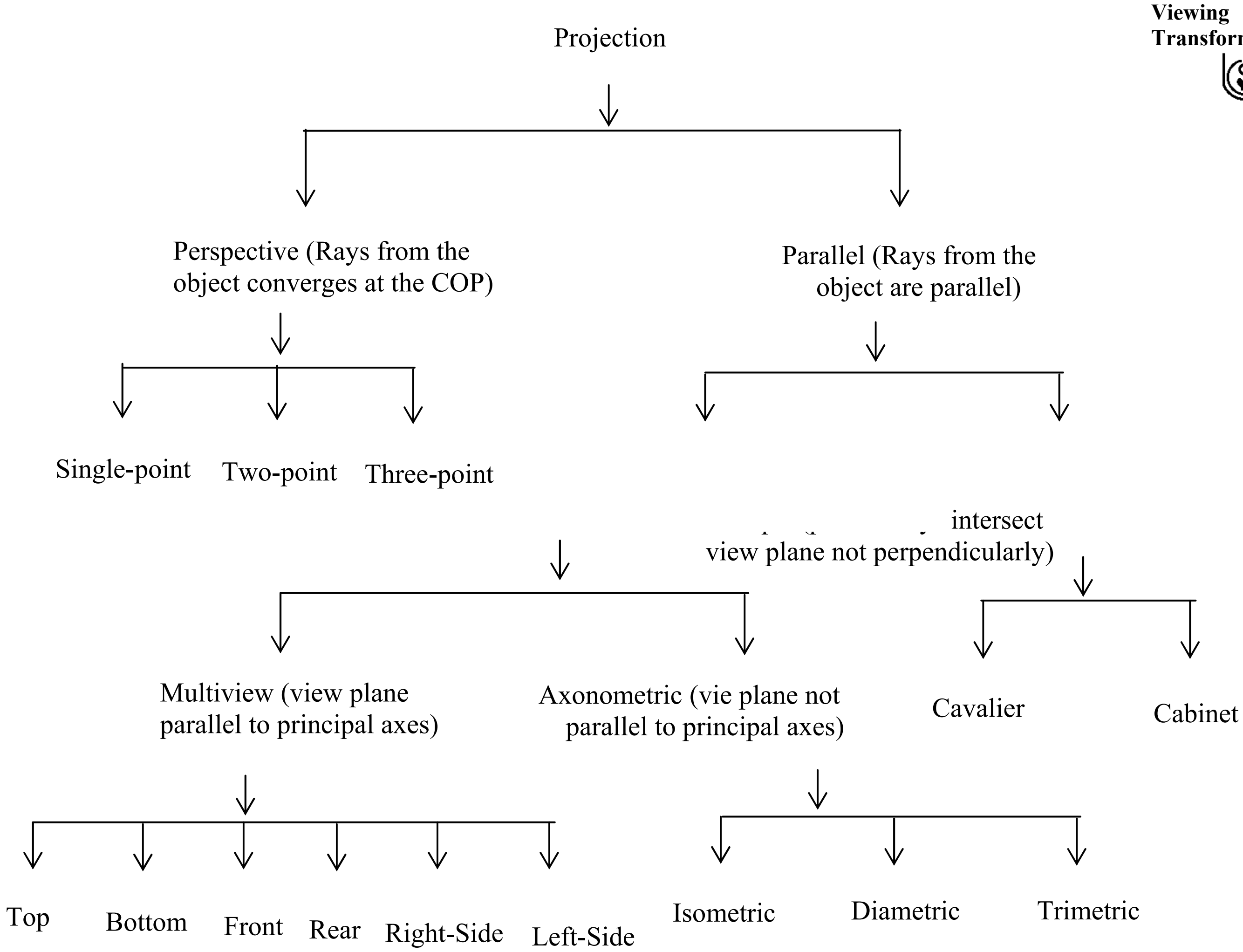


Figure 3: Taxonomy of projection

The direction of rays is very important only in the case of Parallel projection. On the other hand, for Perspective projection, the rays converging at the COP, they do not have a fixed direction i.e., each ray intersects the projection plane with a different angle. For Perspective projection the direction of viewing is important as this only determines the occurrence of a vanishing point.

2.2.1 Parallel Projection

Parallel projection methods are used by engineers to create working drawings of an object which preserves its true shape. In the case of parallel projection the rays from an object converge at infinity, unlike the perspective projection where the rays from an object converse at a finite distance (called COP).

If the distance of COP from the projection plane is infinite then parallel projection (all rays parallel) occurs i.e., when the distance of COP from the projection plane is infinity, then all rays from the object become parallel and will have a direction called "**direction of projection**". It is denoted by **d**=(d1,d2,d3), which means **d** makes unequal/equal angle with the positive side of the x,y,z axes.

Parallel projection can be categorised according to the angle that the direction of projection makes with the projection plane. For example, in Isometric projection, the direction of projection **d**=(d1,d2,d3) makes equal angle (say α) with all the three-principal axes (see *Figure 4*).

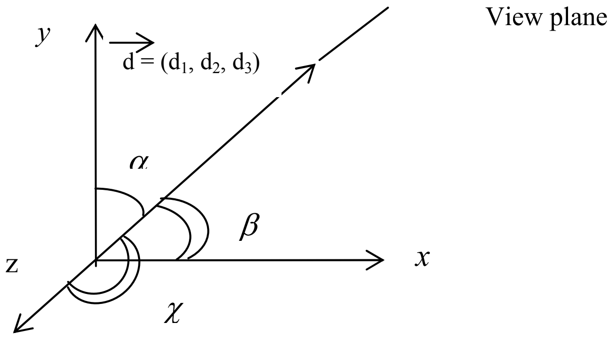


Figure 4: Direction of projection

Rays from the object intersect the plane before passing through COP. In parallel projection, image points are found as the intersection of view plane with a projector (rays) drawn from the object point and having a fixed direction.(see *Figure 5*).

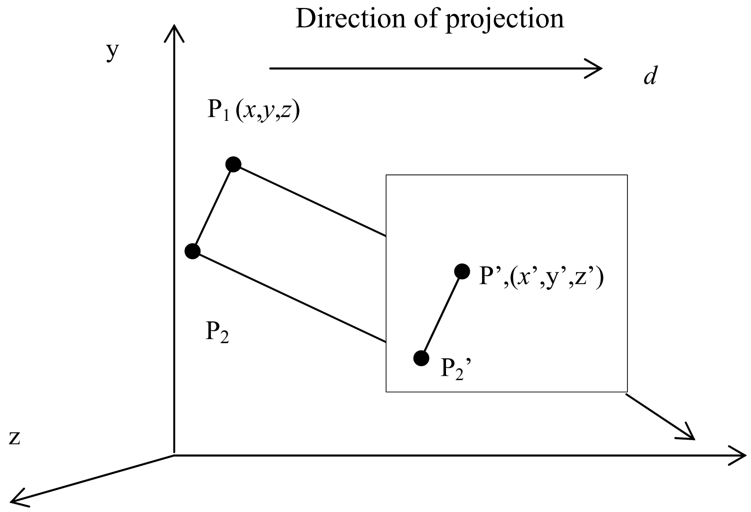


Figure 5: Parallel projection

Parallel rays from the object may be perpendicular or may not be perpendicular to the projection plane. If the direction of projection **d**=(d1,d2,d3) of the rays is perpendicular to the projection plane (or **d** has the same direction as the view plane normal **N**), we have *Orthographic projection* otherwise *Oblique projection.*

Orthographic projection is further divided into *Multiview projection* and *axonometric projection*, depending on whether the direction of projection of rays is parallel to any of the principal axes or not. If the direction of projection is parallel to any of the principal axes then this produces the *front*, *top* and *side views* of a given object, also referred to as *multiview* drawing (see *Figure 8*).

Axonometric projections are orthographic projection in which the direction of projection is not parallel to any of the 3 principle axes. *Oblique* projections are non-orthographic parallel projections i.e., if the direction of projection **d**=(d1,d2,d3) is not perpendicular to the projection plane then the parallel projection is called an *Oblique* projection.

Transformation for parallel projection

Parallel projections (also known as Orthographic projection), are projections onto one of the coordinate planes x = 0, y = 0 or z = 0. The standard transformation for parallel (orthographic) projection onto the xy-plane (i.e. z=0 plane) is:

$P_{par,z} =$

$$\begin{cases} x' = x \\ y' = y \\ z' = 0 \end{cases}$$

In matrix form:

$$P_{par,z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{-----------------------------(1)}$$

Thus, if P(x,y,z) be any object point in space, then projected point P'(x'y'z') can be obtained as:

$$(x', y', z, 1) = (x, y, z, 1) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{-----------------------------(2)}$$

$$P'_h = P_h . P_{par,z} \qquad \text{-----------------------------(3)}$$

Example1: Derive the general transformation of parallel projection onto the xy-plane in the direction of projection d=aI+bJ+cK.

Solution: The general transformation of parallel projection onto the xy-plane in the direction of projection d=a**I**+b**J**+c**K,** is derived as follows(see *Figure a*):

Let P(x,y,z) be an object point, projected to P'(x',y',z') onto the z'=0 plane. From *Figure (a)* we see that the vectors **d** and **PP'** have the same direction. This means that

PP'=k.**d** , comparing components, we have:

x'-x=k.a
y'-y=k.b
z'-z=k.c

Since z'=0 on the projection plane, we get k=-z/c.
Thus,

x'=x-a.z/c
y'=y-b.z/c
z'=0

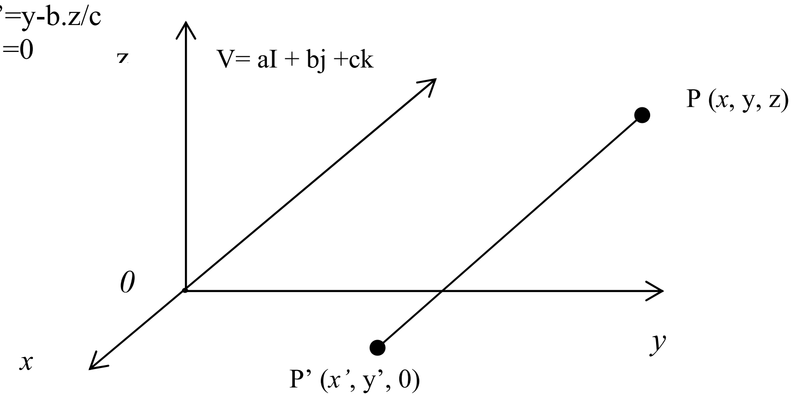


Figure (a)

In terms of homogeneous coordinates, this equation can be written as:

$$(x',y',z'1) = (x, y, z, 1)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -a/c & -b/c & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{-----------------------------(4)}$$

That is, $P'_h = P_h.P_{par,z}$, where $P_{par,z}$ is the parallel projection with the direction of projection **d** along the unit vector k.

Example 2: Derive the general transformation for parallel projection onto a given view plane, where the direction of projection d=aI+bJ+cK is along the normal $N=n_1I+n_2J+n_3K$ with the reference point $R_0(x_0,y_0,z_0)$.

Solution: The general transformation for parallel projection onto the xy-plane in the direction of projection *Figure (b)*
v = a I + bJ + ck, denoted by P par, V, N, Ro, consists of the following steps:

1) Translate the view reference point R_o of the view plane to the origin, by $T\text{-}R_o$
2) Perform an alignment transformation An so that the view normal vector N of the view points in the direction K of the normal to the xy-plane. The direction of projection vector V is transformed to new vector V' = AnV.
3) Project onto the xy-plane using P par, v'
4) Align k back to N, using An.
5) Translate the origin back to Ro, by T_{Ro}

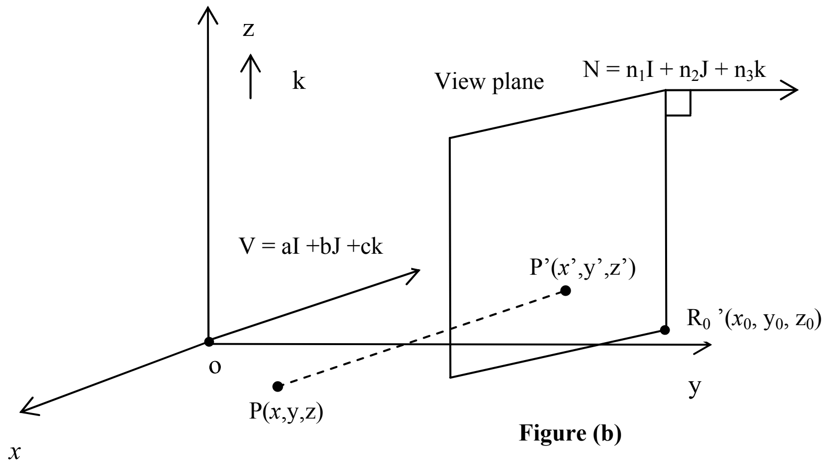


Figure (b)

So
P_{par}, V, N, R_o = T–Ro A_N^{-1} . $P_{par,\,v}$'. An. T_{Ro}

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -x_0 & -y_0 & -z_0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\lambda}{|N|} & \frac{-n_1n_2}{|N|} & \frac{-n_1n_3}{} & 0 \\ 0 & \frac{n_3}{\lambda} & \frac{n_2}{\lambda} & 0 \\ \frac{n_1}{|N|} & \frac{n_2}{|N|} & \frac{n_3}{|N|} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{-a}{c} & 1 & 0 & 0 \\ \frac{-a}{c} & \frac{-b}{c} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\lambda}{|N|} & 0 & \frac{n_2}{|N|} & 0 \\ \frac{-n_1 n_2}{|N|} & \frac{n_3}{\lambda} & \frac{n_2}{|N|} & 0 \\ \frac{-n_1 n_3}{\lambda |N|} & \frac{n_2}{|N|} & \frac{n_3}{|N|} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x_0 & y_0 & z_0 & 1 \end{pmatrix}$$

where λ =

$\sqrt{n_2^2 + n_3^2}$ and $\lambda \neq 0$.

After multiplying all the matrices, we have:

$$P\ par, V, N, R_0 = \begin{pmatrix} d_1 - an_1 & -bn_1 & -cn_1 & 0 \\ -an_2 & d_1 - bn_2 & -cn_2 & 0 \\ -an_3 & -bn_3 & d_1 - cn_3 & 0 \\ ad_0 & bd_0 & c.do & d_1 \end{pmatrix} \text{---------------------------(5)}$$

Where $d_0 = n_1 x_0 + n_2 y_0 + n_3 z_0$ and
$d_1 = n_1 a + n_2 b + n_3 c$

Note: Alignment transformation, An, refer any book for computer graphic.

2.2.1.1 Orthographic and Oblique Projections

Orthographic projection is the simplest form of parallel projection, which is commonly used for engineering drawings. They actually show the 'true' size and shape of a single plane face of a given object.

If the direction of projection **d**=(d1,d2,d3)has the direction of view plane normal **N** (or **d** is perpendicular to view plane**)**, the projection is said to be *orthographic*. Otherwise it is called *Oblique* projection. The *Figure 6* shows the orthographic and oblique projection.

We can see that the orthographic (perpendicular) projection shows only front surface of an object, which includes only two dimensions: length and width. The oblique projection, on the other hand, shows the front surface and the top surface, which includes three dimensions: length, width, and height. Therefore, an oblique projection is one way to show all three dimensions of an object in a single view

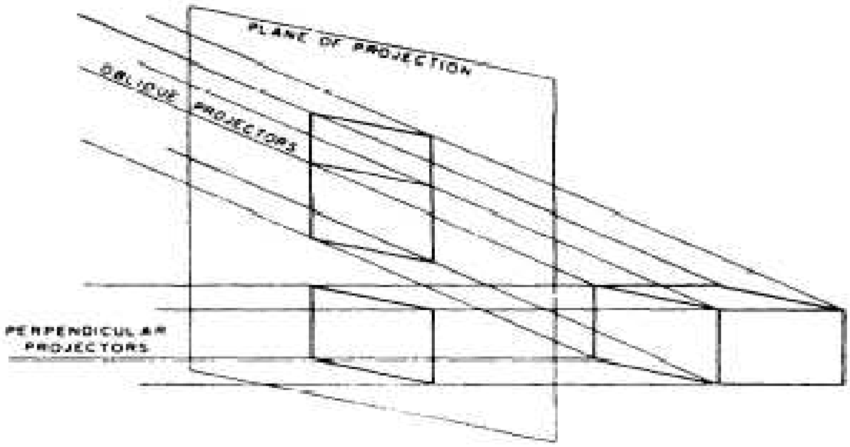


Figure 6: Orthographic and oblique projection

Orthographic projections are projections onto one of the coordinate planes x=0, y=0or z=0. The matrix for orthographic projection onto the z=0 plane (i.e. xy-plane) is:

$$P_{par,z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{------------------------------}(6)$$

Note that the z-column (third column) in this matrix is all zeros. That is for orthographic projection onto the z=0 plane, the z-coordinates of a position vector is set to zero. Similarly, we can also obtain the matrices for orthographic projection onto the x=0 and y=0 planes as:

$$P_{par,x} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{------------------------------}(7)$$

and

$$P_{par,y} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{------------------------------}(8)$$

For example, consider the object given in *Figure 6(a).* The orthographic projections of this object onto the x=0, y=0 and z=0 planes from COP at infinity on the +x-, +y- and +z-axes are shown in *Figure 7* (b)-(d).

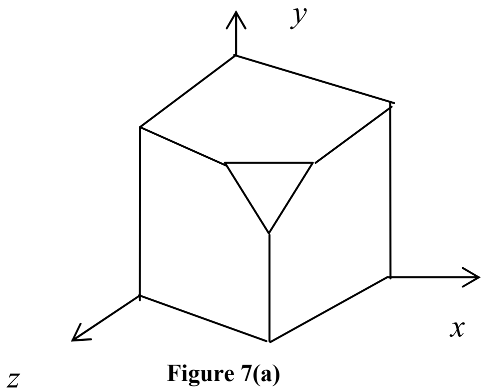


Figure 7(a)

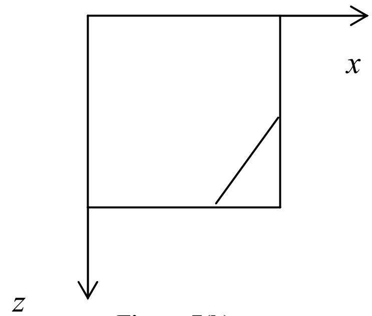


Figure 7(b)

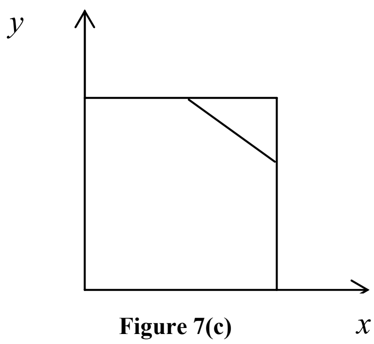


Figure 7(c)

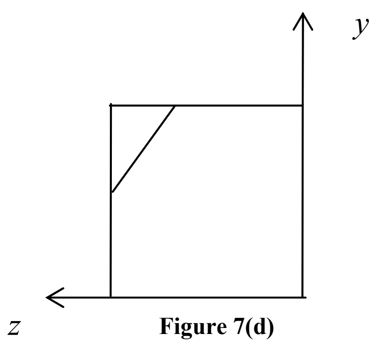


Figure 7(d)

A single orthographic projection does not provide sufficient information to visually and practically reconstruct the shape of an object. Thus multiple orthographic projections are needed (known as *multiview drawing*). In all, we have 6 views:

1) Front view
2) Right-side view
3) Top-view
4) Rear view
5) Left-side view
6) Bottom view

The *Figure 8* shows all 6 views of a given object.

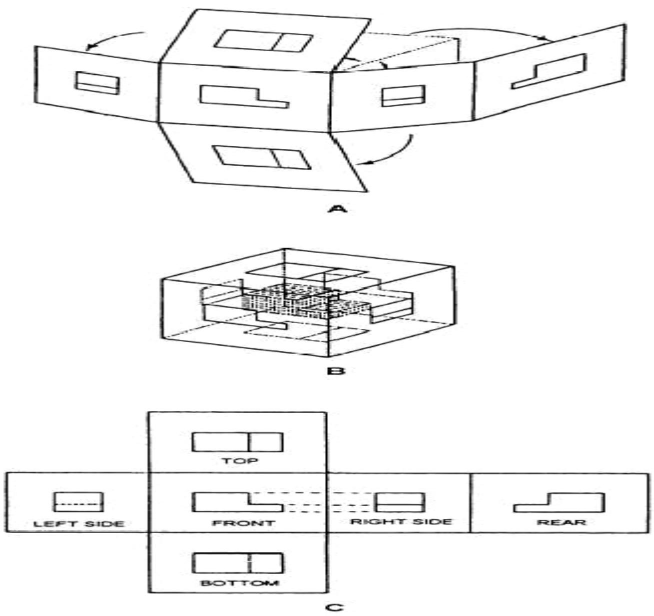


Figure 8: Multiview orthographic projection

The front, right-side and top views are obtained by projection onto the z=0, x=0 and y=0 planes from COP at infinity on the +z-, +x-, and +y-axes.

The rear, left-side and bottom view projections are obtained by projection onto the z=0, x=0, y=0 planes from COP at infinity on the –z-, -x and –y-axes(see *Figure 8*). All six views are normally not required to convey the shape of an object. The front, top and right-side views are most frequently used.

The direction of projection of rays is shown by arrows in *Figure 9*.

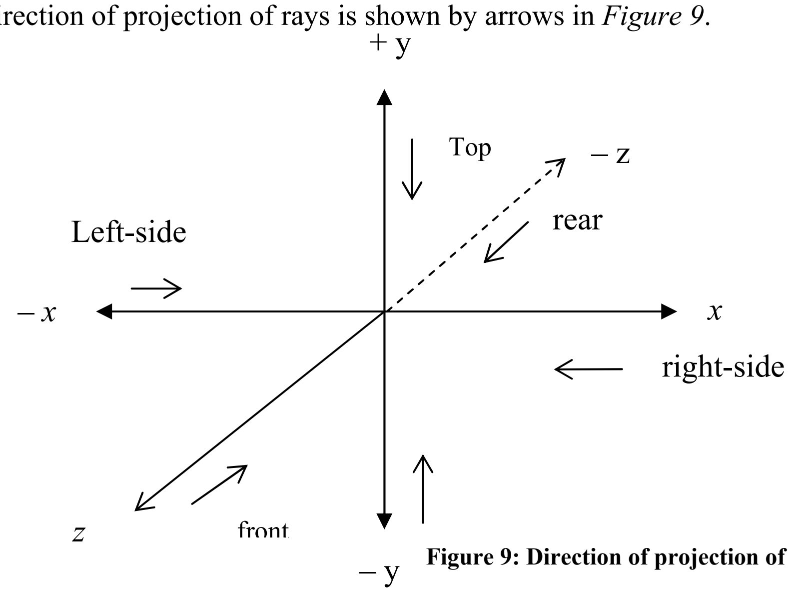


Figure 9: Direction of projection of rays in multiview drawing

The projection matrices for the front, the right-side and top views are given by:

$P_{front} = P_{par,z}=diag(1,1,0,1)$
$P_{right} = P_{par,x}=diag(0,1,1,1)$
$P_{top} = P_{par,y}=diag(1,0,1,1)$

It is important to note that the other remaining views can be obtained by combinations of reflection, rotation and translation followed by projection onto the z=0 plane from the COP at infinity on the +z-axis. For example: the rear view is obtained by reflection through the z=0 plane, followed by projection onto the z=0 plane.

$P_{rear}=M_{xy}.\ P_{par,z}$

$$=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}\text{-------------------------(9)}$$

Similarly, the left-side view is obtained by rotation about the y-axis by $+90^0$, followed by projection onto the z=0 plane.

$P_{left}=[R_y]_{90^0}.P_{par,z}$

$$=\begin{pmatrix}\cos 90&0&-\sin 90&0\\0&1&0&0\\\sin 90&0&\cos 90&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}0&0&0&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix}\text{------------------(10)}$$

And the bottom view is obtained by rotation about the x-axis by -90^0, followed by projection onto the z=0 plane.

$P_{bottom}=[R_x]_{90^0}.P_{par,z}$

$$=\begin{pmatrix}1&0&0&0\\0&\cos(-90)&\sin(-90)&0\\0&-\sin(-90)&\cos(-90)&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\text{-----------(11)}$$

Example 3: Show all the six views of a given object shown in following *Figure*. The vertices of the object are A(4,0,0), B(4,4,0), C(4,4,8), D(4, 0, 4), E (0,0,0), F(0,4,0), G(0,4,8), H(0,0,4).

Solution: We can represent the given object in terms of Homogeneous-coordinates of its vertices as:

$$V=[ABCDEFGH]=\begin{array}{c}A\\B\\C\\D\\E\\F\\G\\H\end{array}\begin{pmatrix}4&0&0&1\\4&4&0&1\\4&4&8&1\\4&0&4&1\\0&0&0&1\\0&4&0&1\\0&4&8&1\\0&0&4&1\end{pmatrix}$$

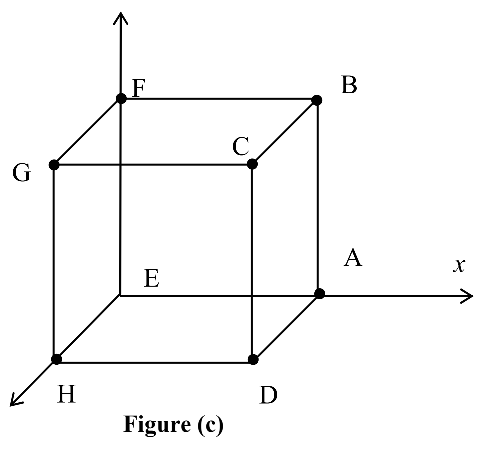


Figure (c)

(1) If we are viewing from the front, then the new coordinate of a given object can be found as:

$P'_{n},z = P_{n}.\ P_{front}$

$$\begin{array}{c}A'\\B'\\C'\\D'\\E'\\F'\\G'\\H'\end{array}\begin{pmatrix} x'1 & y'1 & 1 \\ x'2 & y'2 & 1 \\ x'3 & y'3 & 1 \\ x'4 & y'4 & 1 \\ x'5 & y'5 & 1 \\ x'6 & y'6 & 1 \\ x'7 & y'7 & 1 \\ x'8 & y'8 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 & 1 \\ 4 & 4 & 0 & 1 \\ 4 & 4 & 8 & 1 \\ 4 & 0 & 4 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 4 & 8 & 1 \\ 0 & 4 & 8 & 1 \\ 0 & 0 & 4 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{array}{c}A'\\B'\\C'\\D'\\E'\\F'\\G'\\H'\end{array}\begin{pmatrix} 4 & 0 & 0 & 1 \\ 4 & 4 & 0 & 1 \\ 4 & 4 & 0 & 1 \\ 4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 4 & 0 & 1 \\ 0 & 4 & 0 & 1 \\ 0 & 0 & 0 & \end{pmatrix}$$

from matrix, we can see that
$A' = D'$, $B' = C'$, $E' = H'$, $F' = G'$, Thus we can see only C′D′G′H′
as shown in *Figure d*

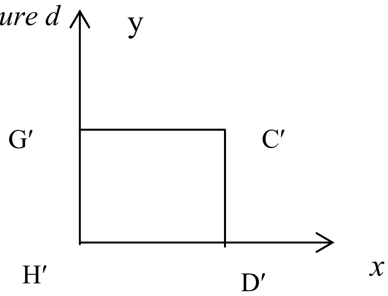


Figure d

(2) If we are viewing from right-side, then

$$P'_{n,x} = V._{\ Pright} = \begin{array}{c}A\\B\\C\\D\\E\\F\\G\\H\end{array}\begin{pmatrix} 4 & 0 & 0 & 1 \\ 4 & 4 & 0 & 1 \\ 4 & 4 & 8 & 1 \\ 4 & 0 & 4 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 4 & 0 & 1 \\ 0 & 4 & 0 & 1 \\ 0 & 0 & 4 & 1 \end{pmatrix} . \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{array}{c}A'\\B'\\C'\\D\\E'\\F'\\G'\\H'\end{array}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 4 & 0 & 1 \\ 0 & 4 & 8 & 1 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 4 & 0 & 1 \\ 0 & 4 & 8 & 1 \\ 0 & 0 & 4 & 1 \end{pmatrix}$$

Here, we can see that A' =E', B' = F', C' = G' and D' = H'.
Thus, we can see only A'B'C'D' as shown in *Figure e*.

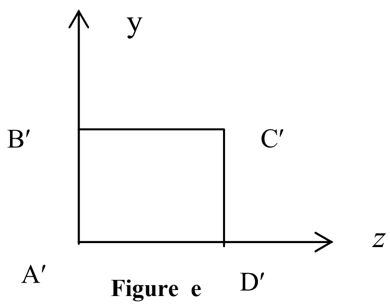


Figure e

(3) if we are viewing from top, then

$$P'_{n,y} = P_n.\ P_{top} = \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{array}\begin{bmatrix} 4 & 0 & 0 & 1 \\ 4 & 4 & 0 & 1 \\ 4 & 4 & 8 & 1 \\ 4 & 0 & 4 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 4 & 0 & 1 \\ 0 & 4 & 8 & 1 \\ 0 & 0 & 4 & 1 \end{bmatrix} . \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{array}{c} A' \\ B' \\ C' \\ D' \\ E' \\ F' \\ G' \\ H' \end{array}\begin{bmatrix} 4 & 0 & 0 & 1 \\ 4 & 0 & 0 & 1 \\ 4 & 0 & 8 & 1 \\ 4 & 0 & 4 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 8 & 1 \\ 0 & 0 & 4 & 1 \end{bmatrix}$$

Here, we can see that A’ = B’, E’ = F’, C’ ≠ D’ and G’ ≠ H’
Thus we can see only the square B’F’G’C’ but the line H’D’ is hidden an shown in *Figure f.*

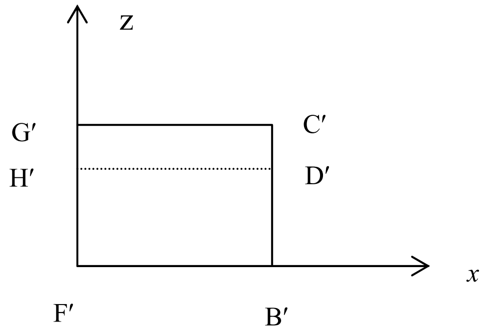


Figure f

Similarly we can also find out the other side views like, rear left-side and bottom using equation – 1, 2, 3

☞ Check Your Progress 1

1) Define the following terms related with Projections with a suitable diagram:

a) Center of Projection (COP)
b) Plane of projection/ view plane
c) Projector
d) Direction of projection

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2) Categories the various types of parallel and perspective projection.

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3) In orthographic projection
   a) Rays intersect the projection plane.
   b) The parallel rays intersect the view plane not perpendicularly.
   c) The parallel rays intersect the view plane perpendicularly.
   d) none of these

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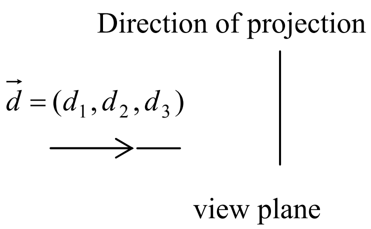
Oblique projection

If the direction of projection **d**=(d1,d2,d3) of the rays is not perpendicular to the view plane(or **d** does not have the same direction as the view plane normal **N**), then the parallel projection is called an *Oblique projection* (see *Figure 10*).



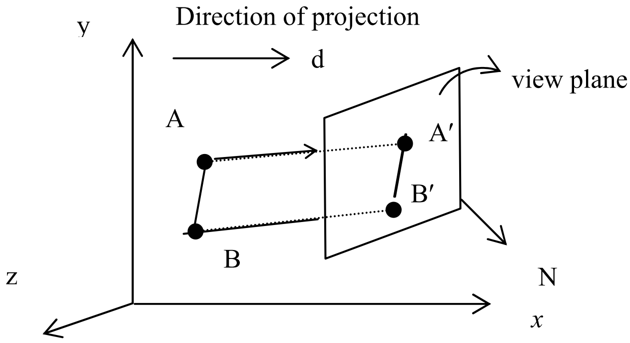


Figure 10 (a): Oblique projection **Figure 10 (b): Oblique projection**

In oblique projection only the faces of the object parallel to the view plane are shown at their true size and shape, angles and lengths are preserved for these faces only. Faces not parallel to the view plane are discarded.

In Oblique projection the line perpendicular to the projection plane are *foreshortened* (shorter in length of actual lines) by the direction of projection of rays. The direction of projection of rays determines the amount of foreshortening. The change in length of the projected line (due to the direction of projection of rays) is measured in terms of foreshortening factor, f.

Foreshortening factors w.r.t. a given direction

Let AB and CD are two given line segments and direction of projection **d**=(d1,d2,d3). Also assumed that AB $\|$ CD $\|$ **d** . Under parallel projection, let AB and CD be projected to A'B' and C'D', respectively.

Observation:

i) A'B' $\|$ C'D' will be true, i.e. Parallel lines are projected to parallel lines, under parallel projection.

ii) |A'B'|/|AB| = |C'D'|/|CD| must be true, under parallel projection.

This ratio (projected length of a line to its true length) is called the foreshortening factor w.r.t. a given direction.

Mathematical description of an Oblique projection (onto xy-plane)

In order to develop the transformation for the oblique projection, consider the *Figure 10*. This *figure* shows an oblique projection of the point A (0, 0, 1) to position A'(x',y',0) on the view plane (z=0 plane). The direction of projection **d=**(d1,d2,d3).

Oblique projections (to xy-plane) can be specified by a number f and an angle θ. The number f, known as foreshortening factor, indicates the ratio of projected length OA'of a line to its true length. Any line L perpendicular to the xy-plane will be foreshortened after projection.

θ is the angle which the projected line OA'(of a given line L perpendicular to xy-plane) makes with the positive x-axis.

The line OA is projected to OA'. The length of the projected line from the origin =|OA'|

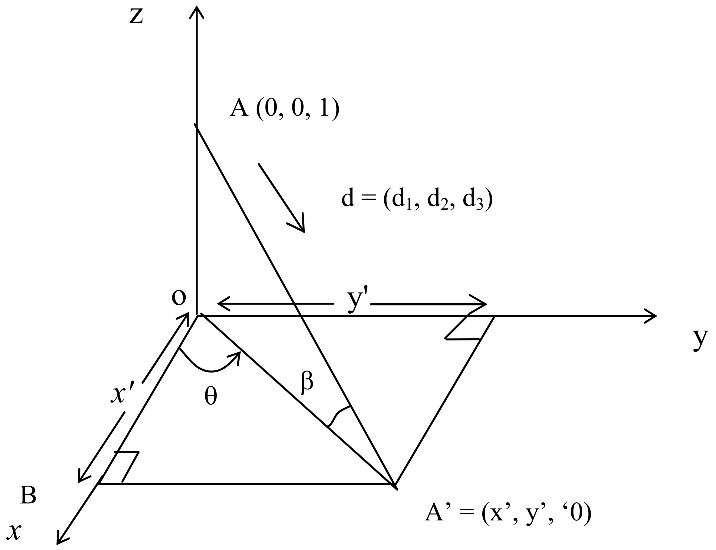


Figure 11: Oblique projection

Thus, foreshortening factor, f=|OA'|/|OA|=|OA'|, in the z-direction
From the triangle OAP', we have,

OB=x'=f.cosθ
BA'=y'=f.sinθ

When f = 1, then obligue projection is known as Cavalier projection

Given θ = 45°, then we have

$$P_{cav} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

When f = ½ then obligue projection is called a cabinet projection.

Here θ = 30° (Given), we have

$$\text{Pcab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \sqrt{3}/4 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

we can represent a given unit cube in terms of Homogeneous coordinates of the

$$\text{vertices as:} V = [A\ B\ C\ D\ E\ F\ G\ H] = \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{array} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

i) To draw the cavalier projection, we find the image coordinates of a given unit cube as follows:

$$P' = V.\ P_{cav} = \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{array} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{array}{c} A' \\ B' \\ C' \\ D' \\ E' \\ F' \\ G' \\ H' \end{array} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ \sqrt{2}/2 & (1+\frac{\sqrt{2}}{2} & 0 & 1 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 & 1 \\ (1+\sqrt{2}/2 & \sqrt{2}/2 & 0 & 1 \\ (1+\sqrt{2}/2 & (1+\sqrt{2}/2 & 0 & 1 \end{bmatrix}$$

Hence, the image coordinate are:

A’ = (0, 0, 0), B’ = (1, 0, 0), C’ = (1, 1, 0), D’ = (0, 1, 0) E’ = ($\sqrt{2}/2$, 1 + $\sqrt{2}/2$, 0)
F’ = ($\sqrt{2}/2$, $\sqrt{2}/2$, 0), G’ = (1 + $\sqrt{2}/2$, $\sqrt{2}/2$, 0), H’ = (1 + $\sqrt{2}/2$, 1 + $\sqrt{2}/2$, 0)

Thus, cavalier projection of a unit cube is shown in *Figure 11(a)*.

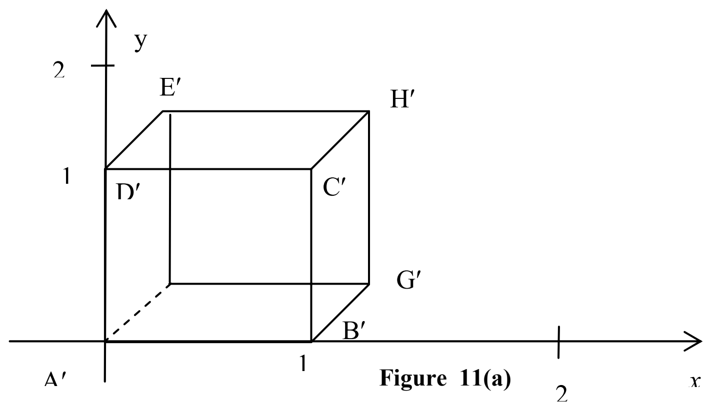


Figure 11(a)

To determine projection matrix for oblique projection, we need to find the direction vector **d.** Since vector **PP'** and vector **d** have the same direction. Thus, **PP'=d**

Thus, $x' - 0 = d_1 = f.\cos\theta$
$y' - 0 = d_2 = f.\sin\theta$
$z' - 1 = d_3$

As $z'=0$ on the xy-plane, $d_3 = -1$,

Since, Oblique projection is a special case of parallel projection, thus, we can transform the general transformation of parallel projection for Oblique projection as follows:

$$P_{Oblique} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -d_1/d_3 & -d_2/d_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ f.\cos\theta & f.\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{-----------------(12)}$$

Where, f=foreshortening factor, i.e., the projected length of the z-axis unit vector. If β is the angle

Between the Oblique projectors and the plane of projection then,
$1/f = \tan(\beta)$, i.e., $f = \cot(\beta)$ ------------- (13)

θ=angle between the projected line with the positive x-axis.

Special cases:

1) If f=0, then $\cot(\beta)=0$ that is $\beta=90^0$, then we have an Orthographic projection.
2) If f=1, the edge perpendicular to projection plane are not foreshortened, then $\beta=\cot^{-1}(1)=45^0$ and this Oblique projection is called *Cavalier* projection.
3) If f=1/2 (the foreshortening is half of unit vector), then $\beta=\cot^{-1}(1/2)=63.435^0$ and this Oblique projection is called *Cabinet* projection.

Note: The common values of θ are **30° and 45°. the values of (180° –** θ) is also acceptable.

The *Figure 12* shows an Oblique projections for foreshortening factor f=1,7/8,3/4,5/8,1/2, with $\theta=45^0$

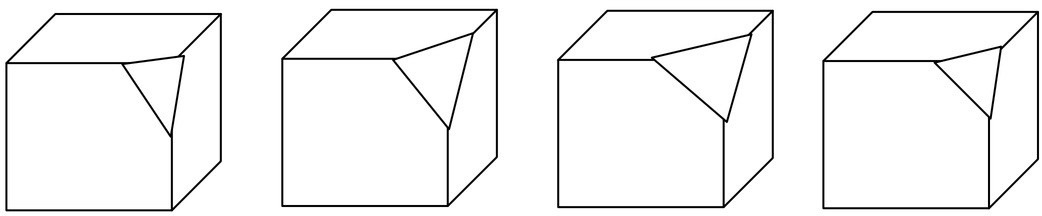


Figure 12: Oblique projections for f=1,7/8,3/4,5/8,1/2, with $\theta=45^0$ (from left to right)

Example4: Find the transformation matrix for a) cavalier projection with $\theta=45^0$, and b) cabinet projection with $\theta=30^0$ c) Draw the projection of unit cube for each transformation.

Solution: We know that cavalier and cabinet projections are a special case of an oblique projection. The transformation matrix for oblique projection is:

$$P_{\text{obligue}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ f.\cos\theta & f.\sin\theta & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(ii) To draw the cabinet projection, we find the image coordinates of a unit cube as:

$$P' \text{ V. Pcab} = \begin{array}{c} A' \\ B' \\ C' \\ D' \\ E' \\ F' \\ G' \\ H' \end{array} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ \sqrt{3}/4 & 5/4 & 0 & 1 \\ \sqrt{3}/4 & 1/4 & 0 & 1 \\ (1+\sqrt{3}/4) & 1/4 & 0 & 0 \\ (1+\sqrt{3}/4) & 5/4 & 0 & 1 \end{bmatrix}$$

Hence, the image coordinates are:
A' (0, 0, 0), B' = (1,0,0), C' = (1, 1, 0), D' = (0, 1, 0), E' = ($\sqrt{3}/4$, 5/4, 0)
F' = ($\sqrt{3}/4$, 1/4, 0), G' = (1 + $\sqrt{3}/4$, 1/4, 0), H' = (1 + $\sqrt{3}/4$, 5/4, 0)

The following *Figure (g)* shows a cabinet projection of a unit cube.

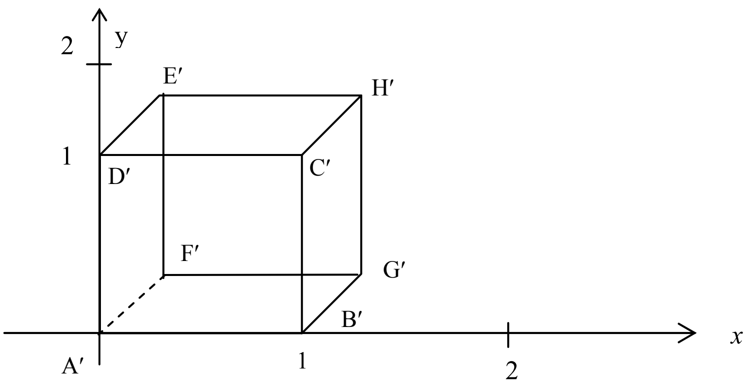


Figure (g)

2.2.1.2 Isometric Projection

There are 3 common sub categories of Orthographic (axonometric) projections:

1) *Isometric*: The direction of projection makes equal angles with all the three principal axes.
2) *Dimetric*: The direction of projection makes equal angles with exactly two of the three principal axes.
3) *Trimetric*: The direction of projection makes unequal angles with all the three principal axes.

Isometric projection is the most frequently used type of *axonometric* projection, which is a method used to show an object in all three dimensions in a single view. Axonometric projection is a form of orthographic projection in which the projectors are always perpendicular to the plane of projection. However, the object itself, rather than the projectors, are at an angle to the plane of projection.

Figure13 shows a cube projected by isometric projection. The cube is angled so that all of its surfaces make the same angle with the plane of projection. As a result, the length of each of the edges shown in the projection is somewhat shorter than the actual length of the edge on the object itself. This reduction is called foreshortening. Since, all of the surfaces make the angle with the plane of projection, the edges foreshorten in the same ratio. Therefore, one scale can be used for the entire layout; hence, the term *isometric* which literally means the same scale.

Construction of an Isometric Projection

In isometric projection, the direction of projection $d = (d_1,d_2,d_3)$ makes an equal angles with all the three principal axes. Let the direction of projection $d = (d_1,d_2,d_3)$ make equal angles (say α) with the positive side of the x,y, and z axes(see *Figure 13*).

Then
$i.d=d_1=|i|.|d|.\cos\alpha \Rightarrow \cos\alpha=d_1/|d|$
similarly
$d_2=j.d=|j|.|d|.\cos\alpha \Rightarrow \cos\alpha=d_2/|d|$
$d_3=k.d=|k|.|d|.\cos\alpha \Rightarrow \cos\alpha=d_3/|d|$

so $\cos\alpha=d_1/|d| = d_2/|d| = d_3/|d|$
⇨ $d_1= d_2 = d_3$ is true
we choose $d_1=d_2=d_3=1$

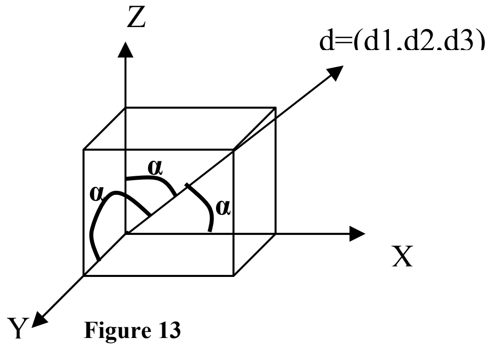


Figure 13

Thus, we have **d =(1, 1, 1)**

Since, the projection, we are looking for is an isometric projection => orthographic projection, i.e, the plane of projection, should be perpendicular to d, so d = n = (1,1,1). Also, we assume that the plane of projection is passing through the origin.

⇨ We know that the equation of a plane passing through reference point $R(x_0,y_0,z_0)$ and having a normal $N = (n_1,n_2,n_3)$ is: $(x - x_0).n_1 + (y - y_0).n_2 + (z - z_0).n_3=0$ -------------------------(14)

Since $(n_1,n_2,n_3)=(1,1,1)$ and
$(x_0,y_0,z_0)=(0,0,0)$
From equation (14), we have $\mathbf{x + y + z = 0}$
Thus, we have the equation of the plane: $\mathbf{x + y + z = 0}$ and **d = (1,1,1)**

Transformation for Isometric projection

Let P(x,y,z) be any point in a space. Suppose a given point P(x,y,z) is projected to P'(x'y',z') onto the projection plane x + y + z = 0. We are interested to find out the projection point P'(x',y',z').

The parametric equation of a line passing through point P(x, y, z) and in the direction of d (1, 1, 1) is:

P + t.**d** = (x, y, z) + t. (1,1,1) = (x + t, y + t, z + t) is any point on the line, where $-\infty< t < \infty$. The point P' can be obtained, when $t = t^*$.
Thus $P'=(x',y',z')=(x + t^*,y + t^*,z + t^*)$, since P' lies on x + y + z = 0 plane.

⇨ $(x + t^*)+(y + t^*) + (z + t^*)=0$
⇨ $3.t^*=-(x + y + z)$
⇨ $t^*=-(x + y + z)/3$ should be true.
⇨ $x'= (2.x - y - z)/3$, $y'=(-x +2.y - z)/3$, $z'=(- x - y +2.z)/3$

Thus, P'=(x',y',z')=[(2.x –y–z)/3, (–x +2.y– z)/3, (-x-y+2.z)/3] --------------------(15)
In terms of homogeneous coordinates, we obtain

$$(x', y', z, 1) = (x, y, z, 1)\begin{pmatrix} 2/3 & -1/3 & 1/3 & 0 \\ -1/3 & 2/3 & -1/3 & 0 \\ -1/3 & -1/3 & 2/3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note: We can also verify this Isometric transformation matrix by checking all the foreshortening factors, i.e., to check whether all the foreshortening factors (fx, fy, fz) are equal or not. Consider the points A,B and C on the coordinate axes (see *Figure14*).

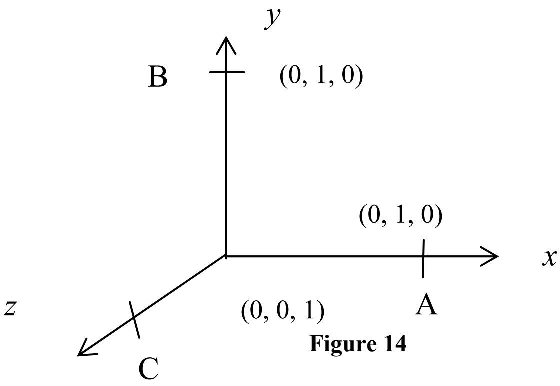


Figure 14

i) Take OA, where O=(0,0,0) and A (1,0,0). Suppose O is projected to O' and A is projected to A'
Thus, by using equation (15), we have O'=(0,0,0) and A'=(2/3,-1/3,-1/3).
So |O'A'| = $\sqrt{(2/3)^2}$ +$(-1/3)^2$+$(-1/3)^2$ = $\sqrt{2/3}$ = fx ------------(16)

ii) Take OB, where O = (0,0,0) and B (0,1,0). Suppose O is projected to O' and B is projected to B'. Thus by using equation (15), we have O'=(0,0,0) and B'= (-1/3,2/3,-1/3).
So |O'B'| = $\sqrt{-(1/3)^2}$ +$(2/3)^2$+$(-1/3)^2$ = $\sqrt{2/3}$ = fy -------------(17)

iii) Take OC, where O=(0,0,0) and C(0,0,1). Suppose O is projected to O' and C is projected to C'

Thus, by using equation(15), we have O'=(0,0,0) and C'=(-1/3,-1/3,2/3).

So |O'C'|=√$(-1/3)^2$+$(-1/3)^2$+$(2/3)^2$ = √2/3=fz --------------(18)

Thus, we have **fx=fy=fz**, which is true for Isometric projection.

Example 5: Obtain the isometric view of a cuboid, shown in *figure*. The size of cuboid is 10x8x6, which is lying at the origin.

Solution: The given cuboids can be represented in terms of Homogeneous coordinates of vertices as shown in *Figure (h):*

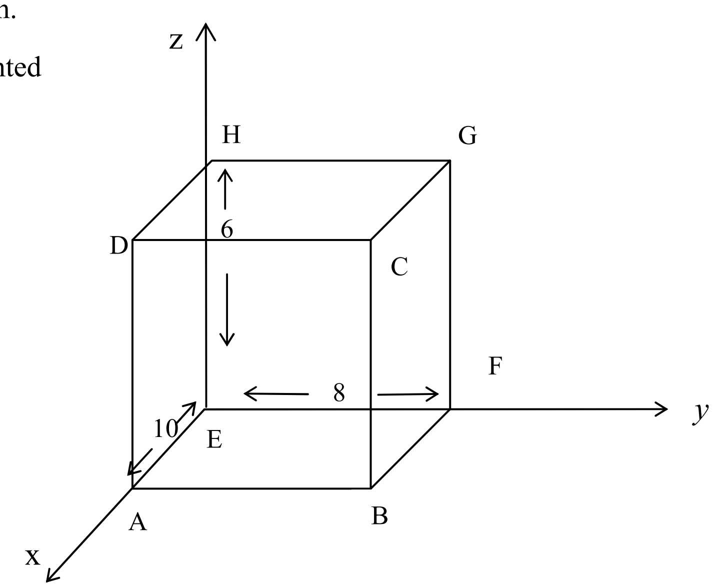


Figure (h)

$$V = [A\ B\ C\ D\ E\ F\ G\ H] = \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{array} \begin{pmatrix} 10 & 0 & 0 & 1 \\ 10 & 8 & 0 & 1 \\ 10 & 8 & 6 & 1 \\ 10 & 8 & 6 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 8 & 0 & 1 \\ 0 & 8 & 6 & 1 \\ 0 & 0 & 6 & 1 \end{pmatrix}$$

To draw an Isometric projection, we find the image coordinate of a given cuboid as follows:

$$P' = V.\,P_{ISO} = \begin{pmatrix} 10 & 0 & 0 & 1 \\ 10 & 8 & 0 & 1 \\ 10 & 8 & 6 & 1 \\ 10 & 0 & 6 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 8 & 0 & 1 \\ 0 & 8 & 6 & 1 \\ 0 & 0 & 6 & 1 \end{pmatrix} . \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} =$$

$$\begin{array}{c} A' \\ B' \\ C' \\ D' \\ E' \\ F' \\ G' \\ H' \end{array} \begin{bmatrix} 20 & -10 & -10 & 3 \\ 12 & 8 & -18 & 3 \\ 6 & 0 & -6 & 3 \\ 14 & -16 & 2 & 3 \\ 0 & 0 & 0 & 3 \\ -8 & 16 & -8 & 3 \\ -14 & 10 & 4 & 3 \\ -6 & -6 & 12 & 3 \end{bmatrix} = \begin{bmatrix} 6.66 & -3.33 & -3.33 & 1 \\ 4.0 & 2.66 & -6.0 & 1 \\ 2 & 0 & -2.0 & 1 \\ 4.66 & -5.33 & 0.66` & 1 \\ 0 & 0 & 0 & 1 \\ -2.66 & 5.33 & 1.33 & 1 \\ -4.66 & 3.33 & 1.33 & 1 \\ -2.0 & -2.0 & 4.0 & 1 \end{bmatrix}$$

Thus, by using this matrix, we can draw an isometric view of a given cuboids.

☞ **Check Your Progress 2**

1) When all the foreshortening factors are different, we have
   a) Isometric b) Diametric c) Trimetric Projection d) All of these.

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2) Distinguish between Orthographic and Oblique parallel projection.

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3) What do you mean by foreshortening factor. Explain Isometric, Diametric and Trimetric projection using foreshortening factors.

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4) Show that for Isometric projection the foreshortening factor along x, y and z-axes must be $\sqrt{2/3}$, i.e. fx = fy = fz = $\sqrt{2/3}$

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5) Consider a parallel projection with the plane of projection having the normal (1,0,–1) and passing through the origin O(0,0,0) and having a direction of projection **d** = (–1,0,0). Is it orthographic projection? Explain your answer with reason.

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6) Compute the cavalier and cabinet projections with angles of 45^0 and 30^0 respectively of a pyramid with a square base of side 4 positioned at the origin in the xy-plane with a height of 10 along the z-axis.

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2.2.2 Perspective Projections

In a perspective projection the center of projection is at finite distance. This projection is called perspective projection because in this projection faraway objects look small and nearer objects look bigger. *See Figure 15 and 16.*

In general, the plane of projection is taken as Z=0 plane.

Properties

1) Faraway objects look smaller.
2) Straight lines are projected to straight lines.
3) Let l_1 and l_2 be two straight lines parallel to each other. If l_1 and l_2 are also parallel to the plane of projection, then the projections of l_1 and l_2 (call them l'_1 and l'_2), will also be parallel to each other.
4) If l_1 and l_2 be two straight lines parallel to each other, but are not parallel to the plane of projection, then the projections of l_1 and l_2 (call them l'_1 and l'_2), will meet in the plane of projection (see *Figure 16*).

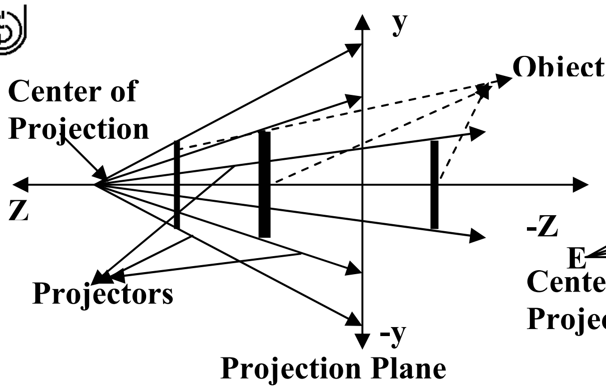

Figure 15

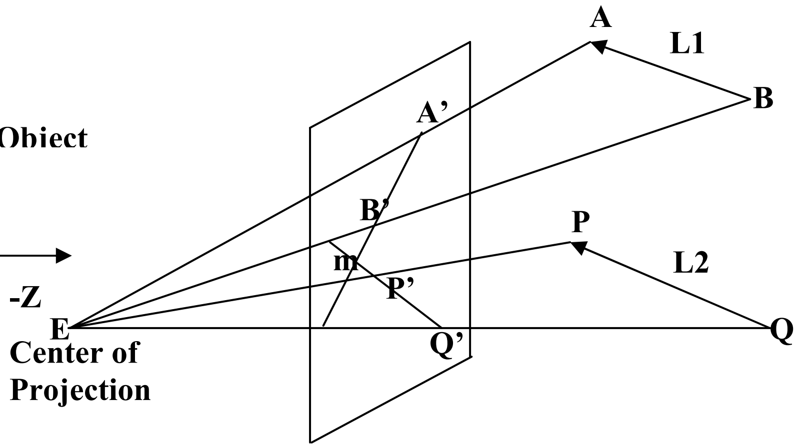

Figure 16

The infinite lines AB and PQ will be projected to the lines A’B’ and P’Q’ respectively on the plane of projection. That is all points of the line AB is projected to all points of the line A’B’. Similarly all points of the line PQ is projected to all points of the line P’Q’. But A’B’ and P’Q’ intersect at M and M is the projection of some point on the line AB as well as on PQ, but AB ∥ PQ, which implies that M is the projection of point at infinity where AB and PQ meet. In this case M is called a **Vanishing point.**

Principle Vanishing point

Suppose l_1 and l_2 be two straight lines parallel to each other, which are also parallel to x-axis. If the projection of l_1 and l_2 (call them l'_1 and l'_2), appears to meet at a point (point at infinity), then the point is called a Principle vanishing point w.r.t. the x-axis. Similarly we have Principle vanishing point w.r.t. the y-axis and z-axis.

Remark

A Perspective projection can have at most 3 Principle Vanishing points and at least one Principle vanishing point.

To understand the effects of a perspective transformation, consider the *Figure 17.* This *figure* shows the perspective transformation on z=0 plane of a given line AB which is parallel to the z-axis. The A*B* is the projected line of the given line AB in the z=0 plane. Let a centre of projection be at (0,0,-d) on the z-axis. The *Figure (A)* shows that the line A’B’ intersects the z=0 plane at the same point as the line AB. It also intersects the z-axis at z=+d. It means the perspective transformation has transformed the intersection point.

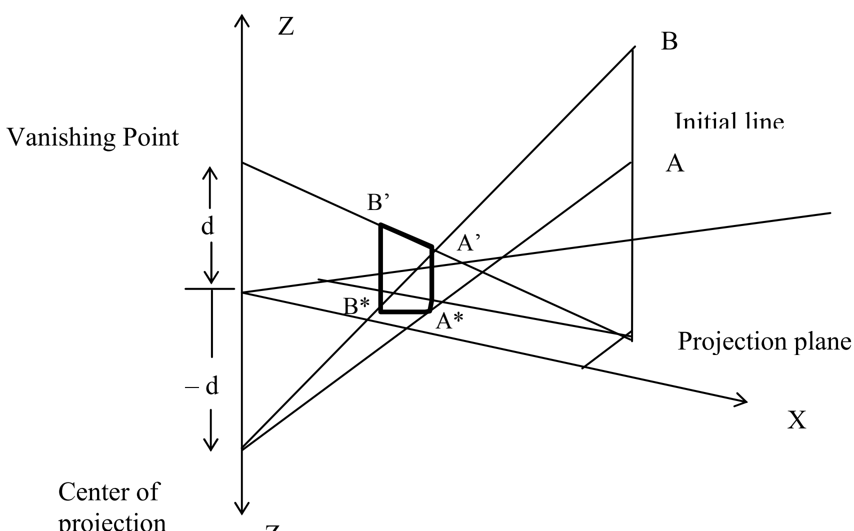

Figure 17

Mathematical description of a Perspective Projection

A perspective transformation is determined by prescribing a C.O.P. and a viewing plane. Let P(x,y,z) be any object point in 3D and C.O.P. is at E(0,0,-d). The problem is to determine the image point coordinates P'(x',y',z') on the Z=0 plane (see *Figure 18)*.

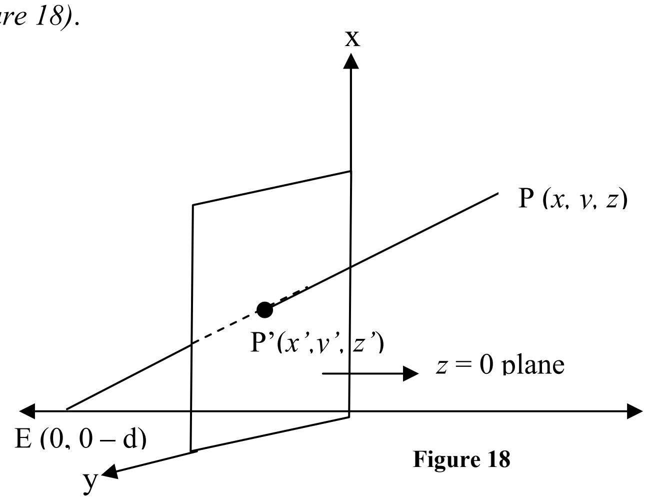

Figure 18

The parametric equation of a line EP, starting from E and passing through P is:

E+t(P-E) 0<t<∞

=(0,0,-d)+t[(x,y,z)-(0,0,-d)]

=(0,0,-d)+t(x,y,z+d)

=[t.x, t.y, -d+t.(z+d)]

Point P' is obtained, when t=t*

That is, P'=(x',y',z') =[t*.x, t*.y, -d+t*.(z+d)]

Since P' lies on Z=0 plane implies -d+t*.(z+d)=0 must be true, that is t*=d/(z+d) is true.

Thus x'=t*.x=x.d/(z+d)
y'=t*.y=y.d/(z+d)
z'=-d+t*(z+d)=0

thus P'=(x.d/(z+d), y.d/(z+d), 0)
=(x/((z/d)+1),y/((z/d)+1),0)
in terms of Homogeneous coordinate system P'=(x,y,0,(z/d)+1). ---------(5)

The above equation can be written in matrix form as:

$$P(x',y',z',1)=(x,y,z,1)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/d \\ 0 & 0 & 0 & 1 \end{pmatrix} = [x,y,0,(z/d)+1] \qquad \text{------------(1)}$$

That is, $P'_h = P_h.P_{per,z}$ ---------------- (2)

Where $P_{per,z}$ in equation (4.6) represents the *single point perspective transformation on z-axis*.

The Ordinary coordinates are:
[x',y',z',1]=[x/(r.z+1),y/(r.z+1),0,1] where r=1/d ---------------- (3)

Vanishing Point

The vanishing point is that point at which parallel lines appear to converge and vanish. A practical example is a long straight railroad track.

To illustrate this concept, consider the *Figure 17* which shows a perspective transformation onto z=0 plane. The *Figure17* shows a Projected line A*B* of given line AB parallel to the z-axis. The center of projection is at (0,0,-d) and z=0 be the projection plane.

Consider the perspective transformation of the point at infinity on the +z-axis, i.e.,

$$[0,0,1,0] \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/d \\ 0 & 0 & 0 & 1 \end{pmatrix} = (0,0,0,1/d) \text{ -------------- } (4)$$

Thus, the ordinary coordinates of a point (x',y',z',1)=(0,0,0,1), corresponding to the transformed point at infinity on the z-axis, is now a finite point. This means that the entire semi-infinite positive space(0<=z<=∞) is transformed to the finite positive half space 0<=z'<=d.

Single point perspective transformation

In order to derive the single point perspective transformations along x and y-axes, we construct *Figures (19) and (20)* similar to *Figure 18*, but with the corresponding COP's at E(-d,0,0) and E(0,-d,0) on the negative x and y-axes respectively.

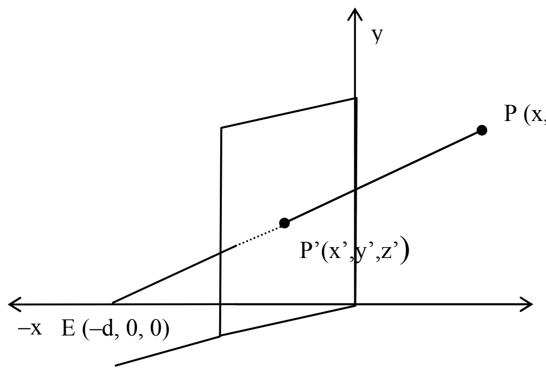

Figure 19

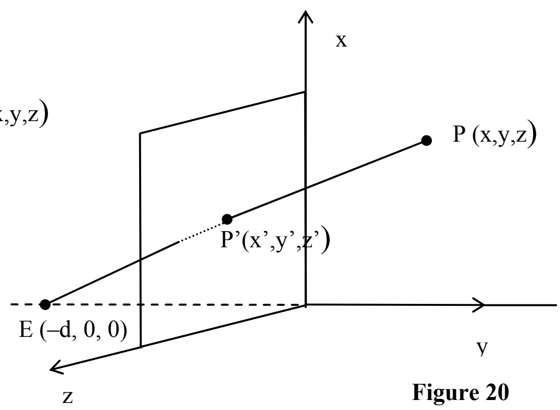

Figure 20

The parametric equation of a line EP, starting from E and passing through P is:

E+t(P-E) 0<t<∞

=(-d,0,0)+t[(x,y,z)-(-d,0,0)]

=(-d,0,0)+t[x+d,y,z]

=[-d+t.(x+d), t.y, t.z]

Point P' is obtained, when t=t*

That is, P'=(x',y',z') =[-d+t*.(x+d), t*.y, t*.z]

Since, P' lies on X=0 plane implies -d+t*.(x+d)=0 must be true, that is t*=d/(x+d) is true.

Thus, x'=-d+t*(x+d)=0
y'=t*.y=y.d/(x+d)
z'=t*.z=z.d/(x+d)
thus P'=(0, y.d/(z+d), z.d/(x+d))
=(0,y/((z/d)+1), z/((x/d)+1))

in terms of Homogeneous coordinate system P'=(0,y,z,(x/d)+1).

The above equation can be written in matrix form as:

$$P(x',y',z',1)=(x,y,z,1)\begin{pmatrix} 0 & 0 & 0 & 1/d \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =[0,y,z,(x/d)+1]$$

=[0,y/((z/d)+1), z/((x/d)+1),1] -------- (5)

That is, $P'_h = P_h.P_{per,x}$ --------------------------------(6)

Where $P_{per,z}$ in equation (5) represents the *single point perspective transformation* w.r.t. x-axis.

Thus, the ordinary coordinates(projected point P' of a given point P) of a *single point perspective transformation* w.r.t. x-axis is:

(x',y',z',1)= [0,y/((z/d)+1), z/((x/d)+1),1] has a center of projection at [-d,0,0,1] and a vanishing point located on the x-axis at [0,0,0,1]

Similarly, the *single point perspective transformation w.r.t. y-axis* is therefore:

$$P(x',y',z',1)=(x,y,z,1)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =[x,0,z,(y/d)+1]$$

=[x/((y/d)+1),0, z/((y/d)+1),1]

That is, $P'_h = P_h.P_{per,y}$ -----------------------------(7)

Where $P_{per,y}$ in equation (5) represents the *single point perspective transformation* w.r.t. y-axis.

Thus, the ordinary coordinates(projected point P' of a given point P) of a *single point perspective transformation* w.r.t. y-axis is:

(x',y',z',1)=[x/((y/d)+1),0, z/((y/d)+1),1] has a center of projection at [0,-d,0,1] and a vanishing point located on the y-axis at [0,0,0,1].

Example 6: Obtain a transformation matrix for perspective projection for a given object projected onto x=3 plane as viewed from (5,0,0).

Solution: Plane of projection: $x = 3$ (given)
Let P (x, y, z) be any point in the space. We know the
Parametric equation of a line AB, starting from A and passing

through B is

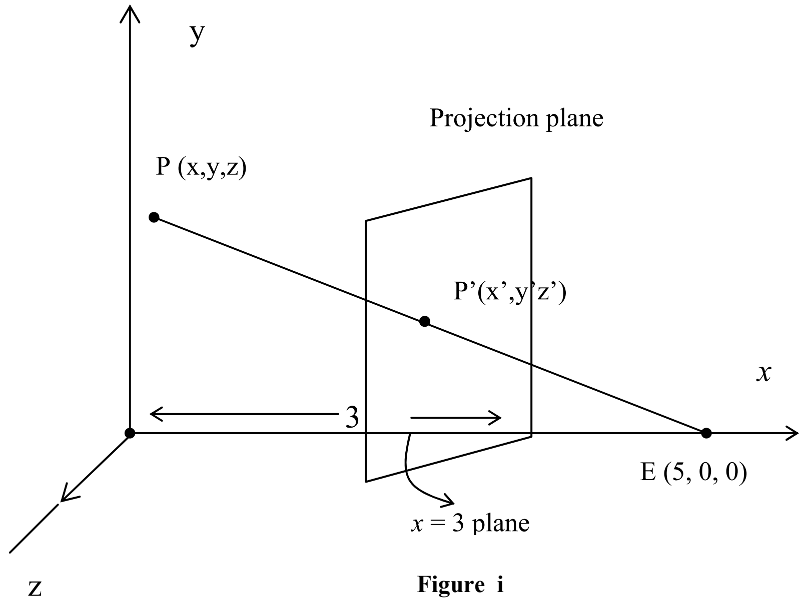

Figure i

P (t) = A + t. (B – A), o < t < ∞

So that parametric equation of a line starting from E (5,0,0) and passing through P (x, y, z) is:

E + t (P – E) , o < t < ∞.
= (5, 0, 0) + t [(*x, y, z*) – (5, 0, 0)]
= (5, 0, 0) + [t (*x* – 5), t. *y*, t. *z*]
= [t. (x – 5) + 5, t. y, t. z]. Assume

Point P' is obtained, when t = t*

∴ P' = (*x*', *y*', *z*') = [t* (*x* – 5) + 5, t**y*, t*. *z*]

Since, P' lies on *x* = 3 plane, so
t* (*x* – 5) + 5 = 3 must be true;

$$t^* = \frac{-2}{x-5}$$

$$P' = (x', y', z') = \left(3, \frac{-2.y}{x-5}, \frac{-2.z}{x-5}\right)$$

$$= \left(\frac{3x-15}{x-5}, \frac{-2.y}{x-5}, \frac{-2.z}{x-5}\right)$$

In Homogeneous coordinate system

$$P' = (x', y', z', 1) = \left(\frac{3x-15}{x-5}, \frac{-2.y}{x-5}, \frac{2.z}{x-5}, 1\right)$$

$$= (3x - 15, -2.y, -2.z, x - 5) \qquad \text{--------------(1)}$$

In Matrix form:

$$(x', y', z', 1) = (x, y, z, 1)\begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ -15 & 0 & 0 & -5 \end{bmatrix} \text{-------------- (2)}$$

Thus, equation (2) is the required transformation matrix for perspective view from (5, 0, 0).

Example 7: Consider the line segment AB in 3D parallel to the z-axis with end points A (– 5,4,2) and B (5,-6,18). Perform a perspective projection on the X=0 plane, where the eye is placed at (10,0,10).

Solution: Let P (x, y, z) be any point in the space.

The parametric equation of a line starting from E and passing through P is:

E + t. (P – E), o < t < 1.
= (10,0,10) + t. [(x, y, z) – (10, 0, 10)]
= (10, 0,10) + t [(x – 10)], y (z – 10)]
= (t. (x – 10) + 10, t. y, t (z – 10) + 10)

Assume point P' can be obtained, when t = t*

$\therefore$ P' = (x', y', z') = (t* (x – 10) + 10, t*.y, t*. (z – 10) + 10)

since point P' lies on x = 0 plane

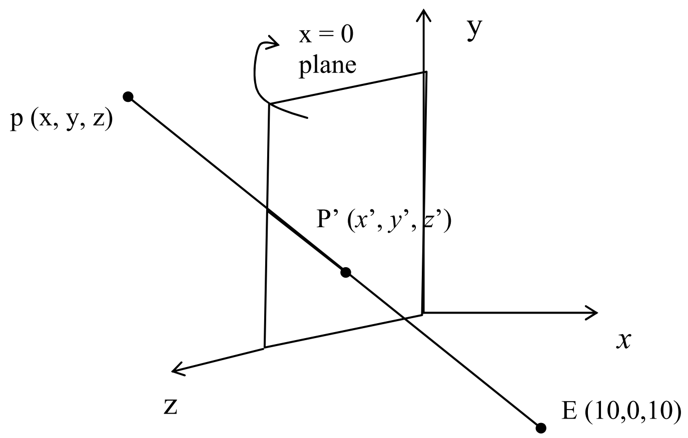

Figure j

= t* (x – 10) + 10 = 0

$$= t^* = \frac{-10}{x-10}$$

$$= P' = (x', y', z') = \left(0, \frac{-10.y}{x-10}, \frac{-10(z-10)}{x-10} + 10\right)$$

$$\left(0, \frac{-10.y}{x-10}, \frac{10.x-10.z}{x-10}\right)$$

In terms of Homogeneous coordinate system;

$$P' = (x', y', z', 1) = \left(0, \frac{-y}{\frac{x}{10}-1}, \frac{x-z}{\frac{x}{10}-1}, 1\right) = (0, -y, x-z, \frac{x}{10}-1)$$

In Matrix form

$$(x', y', z', 1) = (x, y, z, 1)\begin{bmatrix} 0 & 0 & 1 & 1/10 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \text{------------------(1)}$$

This equation (1) is the required perspective transformation, which gives a coordinates of a projected point P' (x', y', z') onto the $x = 0$ plane, when a point p (x, y, z) is viewed from E (10, 0, 10)

Now, for the given points A (–5, 4, 2) and B (5, –6, 18), A' and B' are their projection on the $x = 0$ plane.

Then from Equation (1).

$$A' = (x'_1, y'_1, z'_1, 1) = (-5, 4, 2, 1).\begin{bmatrix} 0 & 0 & 1 & 1/10 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$= (0, -4, -7, \frac{-5}{10} - 1)$$

$$= (0, -4, -7, \frac{-15}{10})$$

$$= (0, -40, -70, -15)$$

$$= (0, \frac{40}{15}, \frac{70}{15}, 1)$$

Hence $x_1' = 0$; $y_1' = 2.67$; $z_1' = 4.67$

similarly $$B' = (x_2', y_2', z_2', 1) = (5, -6, 18, 1) . \begin{bmatrix} 0 & 0 & 1 & 1/10 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$= (0, 60, -130, -5)$$

$$= (0, -12, 26, 1)$$

Hence $x_2' = 0$; $y_2' = -12$; $z_2' = 26$

Thus the projected points A' and B' of a given points A and B are:

$$A' = (x_1', y_1' z_1') = (0, 2.67, 4.67) \quad \text{and} \quad B' = (x_2', y_2', z_2') = (0, -12, 26, 1)$$

Example 8: Consider the line segment AB in *Figure k*, parallel to the z-axis with end points A (3, 2, 4) and B (3, 2, 8). Perform a perspective projection onto the z = 0 plane from the center of projection at E (0, 0, – 2). Also find out the vanishing point.

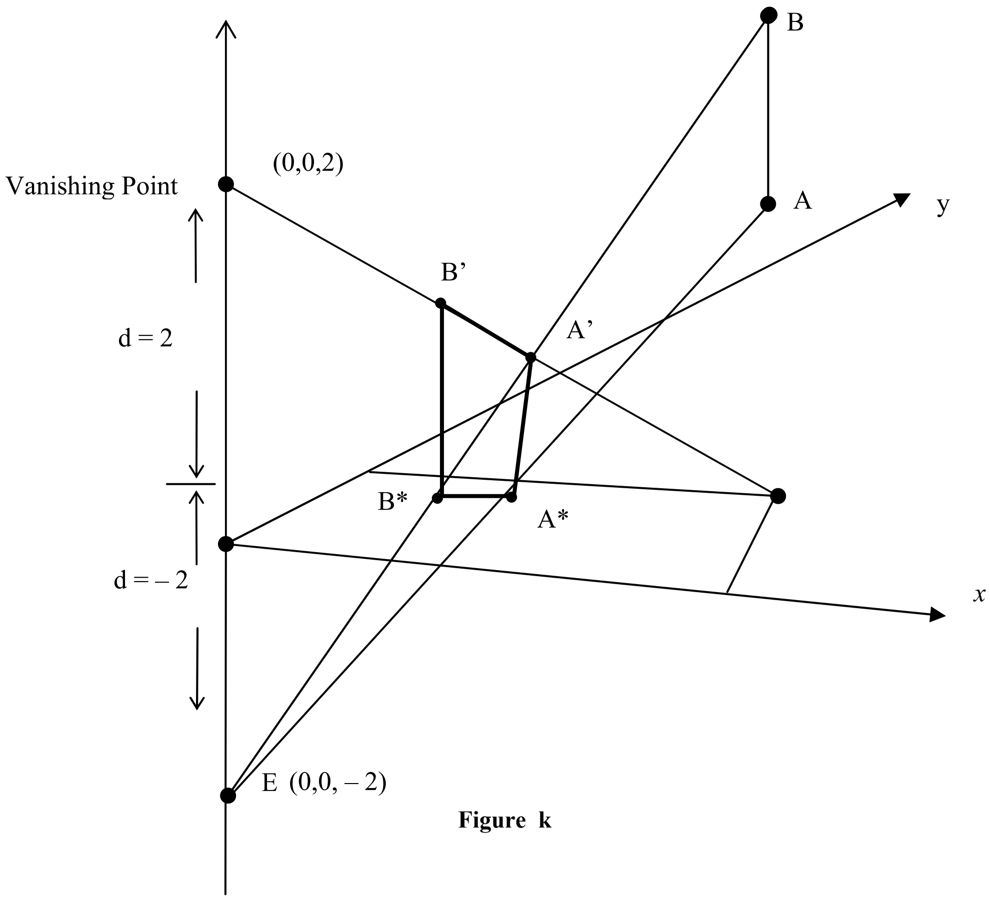

Figure k

Solution. We know that (from Equation (1)), the center of single point perspective transformation: of a point P (x, y, z) onto z = 0 plane, where center of projection is at (0, 0, – d) is given by:

$$(x', y', z', 1) = (x, y, z, 1) \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/\mathrm{d} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P'n = Pn.\ P_{per,z} \quad \text{------------------------(I)}$$

Thus the perspective transformation of a given line AB to A* B* with d = 2 is given by:

$$V'_n = V_n.\ P_{per,z}$$

$$\begin{array}{c} A^* \\ B^* \end{array} \begin{bmatrix} x_1^* & y_1^* & z_1^* & 1 \\ x_1^* & y_2^* & z_1^* & 1 \end{bmatrix} = \begin{array}{c} A \\ B \end{array} \begin{bmatrix} 3 & 2 & 4 & 1 \\ 3 & 2 & 8 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{array}{c} A^* \\ B^* \end{array} \begin{bmatrix} 1 & 0.667 & 0 & 1 \\ 0.6 & 0.4 & 0 & 1 \end{bmatrix}$$

Hence, the projected points of a given line AB is:

A* = (1, 0.667, 0)
B* = (0.6, 0.4, 0)

The vanishing point is (0, 0, 0).

Example 9: Perform a perspective projection onto the z = 0 plane of the unit cube, shown in *Figure (l)* from the cop at E (0, 0, 10) on the z-axis.

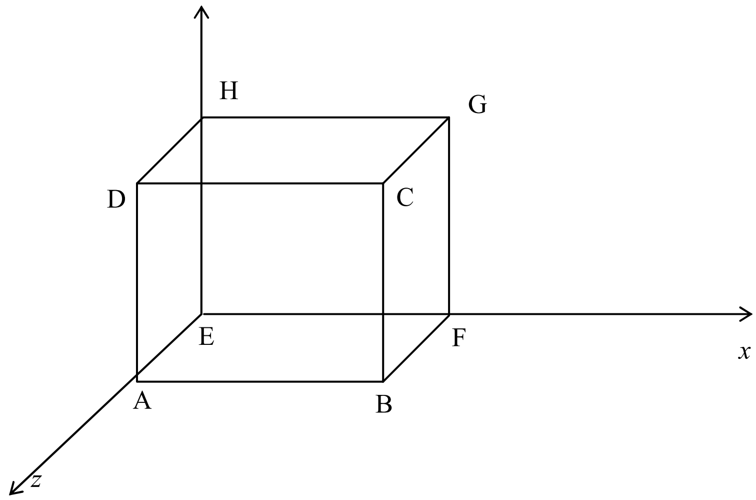

Figure (l)

01: Here center of projection

E = (0, 0, –d) = (0, 0, 10).

$\therefore\ d = -10$

we know that (from equation – 1), the single point perspective transformation of the projection with $z = 0$, plane, where cop is at (0, 0, –d) is given by:

$$(x', y', z', 1) = (x, y, z, 1)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/d \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{-----------(I)}$$

$$P_n' = P.\ P_{per,\,z} \qquad \text{--------------- (II)}$$

Thus the perspective transformation of a given cube v = [ABCDEFGH] to V' = [A'B'C'D'E'F'G'H'] with d = – 10 is given by:

$$[V'] = [V]\ .\ [P_{per,\,z}]$$

$$= \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$V' = \begin{bmatrix} 0 & 0 & 0 & 0.9 \\ 1 & 0 & 0 & 0.9 \\ 1 & 1 & 0 & 0.9 \\ 0 & 1 & 0 & 0.9 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{array}{c} A' \\ B' \\ C' \\ D' \\ E' \\ F' \\ G' \\ H' \end{array} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1.11 & 0 & 0 & 1 \\ 1.11 & 1.11 & 0 & 1 \\ 0 & 1.11 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Thus the projected points of a given cube V = [ABCDEFGH] are:
A' =(0, 0, 0), B' = (1.11, 0, 0), C' = (1.11, 1.11, 0), D' = (0, 1.11, 0), E' = (0, 0, 0)
F' = (1, 0, 0), G' = (1, 1, 0) and H' = (0, 1, 0).

☞ Check Your Progress 3

1) Obtain the perspective transformation onto z = d plane, where the c. o. p. is at the origin.

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2) Consider a cube given in example – 4, the cube can be centered on the z-axis by translating it $-\frac{1}{2}$ units in the x y directions perform a single point perspective transformation onto the z = 0 plane, with c. o. p. at Zc = 10 on the z-axis.

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3) A unit cube is placed at the origin such that its 3-edges are lying along the x, y and z-axes. The cube is rotated about the y-axis by 30°. Obtain the perspective projection of the cube viewed from (80, 0, 60) on the z = 0 plane.

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Two-Point and Three-Point Perspective transformations

The 2-point perspective projection can be obtained by rotating about one of the principal axis only and projecting on X=0 (or Y=0 or Z=0) plane. To discuss the phenomenon practically consider an example for 3-point perspective projection (given below) some can be done for 2-point aspect.

Example 10: Find the principal vanishing points, when the object is first rotated w.r.t. the y-axis by – 30° and x-axis by 45°, and projected onto z = 0 plane, with the center of projection being (0, 0, – 5).

Solution: Rotation about the y-axis with angle of rotation
$\theta = (-30°)$ is

$$[R_y] = [Ry]_{\theta = -30} = \begin{bmatrix} \cos(30°) & 0 & -\sin(-30°) \\ 0 & 1 & 0 \\ \sin(-30°) & 0 & \cos(-30°) \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{3}/2 \end{bmatrix}$$

Similarly Rotation about the x-axis with angle of Rotation ϕ 45° is:

$$[R_x] = [R_x]_{45°} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\therefore\ [R_y].[R_x] = \begin{bmatrix} \sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{3}/2 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{3}/2 & -1/2\sqrt{2} & 1/2\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ -1/2 & -3/2\sqrt{2} & \sqrt{3}/2\sqrt{2} \end{bmatrix} \text{-----------------(1)}$$

Projection: Center of projection is E (0, 0, – 5) and plane of projection is z = 0 plane.

For any point p (x, y, z) from the object, the Equation of the ray starting from E and passing through the point P is:

$$E + t\,(P - E),\ t > 0$$

i.e. $(0, 0, -5) + t\,[(x, y, z) - (0, 0, -5)]$
$= (0, 0, -5) + t\,(x, y, z + 5)$
$= (t.x,\ t.\,y,\ -5 + t\,(z + 5))$

for this point to be lie on z = 0 plane, we have:
$-5 + t\,(z + 5) = 0$

$$\therefore\ t = \frac{5}{z+5}$$

$\therefore$ the projection point of p (x, y, z) will be:

$$P' = (x', y', z') = \left(\frac{5.x}{z+5}, \frac{5.y}{z+5}, 0\right)$$

In terms of homogeneous coordinates, the projection matrix will become:

$$[P] = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix} \text{-------------(2)}$$

$$\therefore [R_y].[R_x].[P] = \begin{bmatrix} \sqrt{3}/2 & -1/2\sqrt{2} & 1/2\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1\sqrt{2} & -\sqrt{3}/2\sqrt{2} & \sqrt{3}/2\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5\sqrt{3}}{2} & \frac{-5}{2\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} \\ 0 & \frac{5}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-5}{2} & \frac{-5\sqrt{3}}{2\sqrt{2}} & 0 & \frac{\sqrt{3}}{2\sqrt{2}} \\ 0 & 0 & 0 & 5 \end{bmatrix} \quad \text{----------------(3)}$$

Let (x, y, z) be projected, under the combined transformation (3) to (x', y', z'), then

$$(x', y', z', 1) = (x, y, z, 1) \begin{bmatrix} \frac{5\sqrt{3}}{2} & \frac{-5}{2\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} \\ 0 & \frac{5}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{-5}{2} & \frac{-5\sqrt{3}}{2\sqrt{2}} & 0 & \frac{\sqrt{3}}{2\sqrt{2}} \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$= x' = \frac{\left(\frac{5\sqrt{3}}{2}.x - \frac{5}{2}.z\right)}{\left(\frac{x}{2\sqrt{2}} + \frac{y}{\sqrt{2}} + \frac{\sqrt{3}.z}{2\sqrt{2}} + 5\right)}$$

and

$$\left. y' = \frac{\left(\frac{-5}{2\sqrt{2}}.x + \frac{5}{\sqrt{2}}.y - \frac{5\sqrt{3}}{2\sqrt{2}}.z\right)}{\left(\frac{x}{2\sqrt{2}} + \frac{y}{\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}.z + 5\right)} \right\} \quad \text{-----------(4)}$$

Case 1: Principal vanishing point w.r.t the x-axis.

By considering first row of the matrix (Equation – (3)), we can claim that the principal vanishing point (w.r.t) the x-axis) will be:

$$\left(\frac{\frac{5\sqrt{3}}{2}}{\frac{1}{2\sqrt{2}}}, \frac{\frac{-5}{2\sqrt{2}}}{\frac{1}{2\sqrt{2}}}, 0 \right)$$

i.e., $(5\sqrt{6}, -5, 0)$ -------------------(I)

In order to varify our claim, consider the line segments AB, CD, which are parallel to the x-axis, where A = (0, 0, 0), B = (1, 0, 0), C = (1, 1, 0), D = (0, 1, 0)

If A’, B’, C’, D’ are the projections of A, B, C, D, respectively, under the projection matrix (3), then

$$A' = (0, 0, 0),\ B' = \left(\frac{5\sqrt{3}}{\frac{1}{2\sqrt{2}}+5}, \frac{\frac{-5}{2\sqrt{2}}}{\frac{1}{2\sqrt{2}}+5}, 0 \right)$$

$$C' = \left(\frac{\frac{5\sqrt{3}}{2}}{\left(\frac{1}{2\sqrt{2}}+\frac{1}{\sqrt{2}}+5\right)}, \frac{\left(-\frac{5}{2\sqrt{2}}+\frac{5}{\sqrt{2}}\right)}{\left(\frac{1}{2\sqrt{2}}+\frac{1}{\sqrt{2}}+5\right)}, 0 \right)$$

$$D' = \left(0, \frac{5/\sqrt{2}}{\left(\frac{1}{\sqrt{2}}+5\right)}, 0 \right) \qquad \{\text{Using Equation (4)}\}$$

$$A' = (0,0,0)\ ,\ B' = \left(\frac{5\sqrt{6}}{1+10\sqrt{2}}, \frac{-5}{1+10\sqrt{2}}, 0 \right),$$

$$C' = \left(\frac{5\sqrt{6}}{3+10\sqrt{2}}, \frac{5}{3+10\sqrt{2}}, 0 \right) \text{ and}$$

$$D' = \left(0, \frac{5}{1+5\sqrt{2}}, 0 \right)$$

Consider the line equation of A'B': The parametric Equation is:

$$A' + t\,(B' - A')$$

i.e. $(0, 0, 0) + t \left(\frac{5\sqrt{6}}{1+10\sqrt{2}}, \frac{-5}{1+10\sqrt{2}}, 0 \right)$

$$= \left(\frac{5t\sqrt{6}}{1+10\sqrt{2}}, \frac{-5.t}{1+10\sqrt{2}}, 0 \right)$$

we will verify that the vanishing point (I) lies on this line:

i.e. $\left(\frac{5t\sqrt{6}}{1+10\sqrt{2}}, \frac{-5.t}{1+10\sqrt{2}}, 0 \right) = (5\sqrt{6}, -5, 0)$

$$= \frac{5.t.\sqrt{6}}{1+10\sqrt{2}} = 5\sqrt{6}$$

and $\frac{-5t}{1+10\sqrt{2}} = -5$ -----------------(5)

must be true for some 't' value.

$t = (1 + 10\sqrt{2})$

then the equation (5) is true and hence (I) lies on the line A'B'.
Similarly consider the line equation C'D': The parametric Equation is:

$$C' + s\,(D' - C') \qquad \text{i.e.}$$

$$= \left(\frac{5\sqrt{6}}{3+10\sqrt{2}}, \frac{5}{3+10\sqrt{2}}, 0\right) + s\left[\left(0, \frac{5}{1+5\sqrt{2}}, 0\right) - \left(\frac{5\sqrt{6}}{3+10\sqrt{2}}, \frac{5}{3+10\sqrt{2}} 0\right)\right]$$

$$= \left(\frac{5\sqrt{6}}{3+10\sqrt{2}}, \frac{5}{3+10\sqrt{2}}, 0\right) + s\left(\frac{-5\sqrt{6}}{3+10\sqrt{2}}, \frac{5}{1+5\sqrt{2}} - \frac{5}{3+10\sqrt{2}}, 0\right) \text{ and}$$

$$= \left(\frac{5\sqrt{6}}{3+10\sqrt{2}} \frac{-5.s.\sqrt{6}}{3+10\sqrt{2}}, \frac{5}{3+10\sqrt{2}} + \frac{5.s.(2+5\sqrt{2})}{(1+5\sqrt{2})(3+10\sqrt{2})}, 0\right), \text{ but}$$

we have to verify that the vanishing point (I) lies on C'D'.

i.e. we have to show

$$\left(\frac{5\sqrt{6}}{3+10\sqrt{2}}(1-s)\frac{5}{3+10\sqrt{2}}\left(1+\frac{s(2+5\sqrt{2})}{(1+5\sqrt{2})}\right), 0\right) = (5\sqrt{6}, -5, 0)$$

for some 's' value This holds true if

$$\left.\begin{aligned} &\frac{5\sqrt{6}}{3+10\sqrt{2}}(1-s) = 5\sqrt{6} \\ \text{and } &\frac{5}{3+10\sqrt{2}}\left(1+\frac{s(2+5\sqrt{2})}{(1+5\sqrt{2})}\right) = -5 \end{aligned}\right\} \qquad \text{------------(6)}$$

must holds simultaneously for some 's' value.

If we choose s = –2 (1 + 5 $\sqrt{2}$), then both the conditions of (6) satisfied
$\therefore$ (5 $\sqrt{6}$, –5, 0) lies on C'D'

= (5 $\sqrt{6}$, – 5, 0) is the point at intersection of A'B' and C'D'.
(5 $\sqrt{6}$, – 5, 0) is the principal vanishing point w.r.t. the *x*-axis.

Case 2: Principal vanishing point w.r.t *y*-axis:-

From the 2nd Row of the matrix (Equation (3)), the principal vanishing point w.r.t *y*-axis will be:

$$\left(0, \frac{5}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \text{ in homogeneous system.}$$

The vanishing point in Cartesian system is:

$$\left(0, \frac{5/\sqrt{2}}{1/\sqrt{2}}, 0\right) = (0, 5, 0) \qquad \text{------------------------(II)}$$

similar proof can be made to verify our claim:

Case 3: Principal vanishing point w.r.t z-axis:

From the 3rd row of matrix equation (3), we claim that the principal vanishing point w.r.t z-axis will be: $\left(-\frac{5}{2}, \frac{-5\sqrt{3}}{2\sqrt{2}}, 0, \frac{\sqrt{3}}{2\sqrt{2}}\right)$ in Homogeneous system.

In Cartesian system, the vanishing point is:

$$\left(\frac{(-5/2)}{\frac{\sqrt{3}}{2\sqrt{2}}}, \frac{\left(\frac{-5}{2}\frac{\sqrt{3}}{\sqrt{2}}\right)}{\left(\frac{\sqrt{3}}{2\sqrt{2}}\right)}, 0\right) = \left(\frac{-5\sqrt{2}}{\sqrt{3}}, -5,\ 0\right) \qquad \text{------------------(III)}$$

A similar proof can be made to verify (III)

General Perspective transformation with COP at the origin

Let the given point P(x,y,z) be projected as P'(x',y',z') onto the plane of projection. The COP is at the origin, denoted by O(0,0,0). Suppose the plane of projection defined by the normal vector **N**=n_1**I**+n_2**J**+n_3**K** and passing through the reference point $R_0(x_0,y_0,z_0)$. From *Figure 21*, the vectors **PO** and **P'O** have the same direction. The vector **P'O** is a factor of **PO**. Hence they are related by the equation: **P'O** = α **PO**, comparing components we have x'=α.x y'=α.y z'=α.z we now find the value of α.

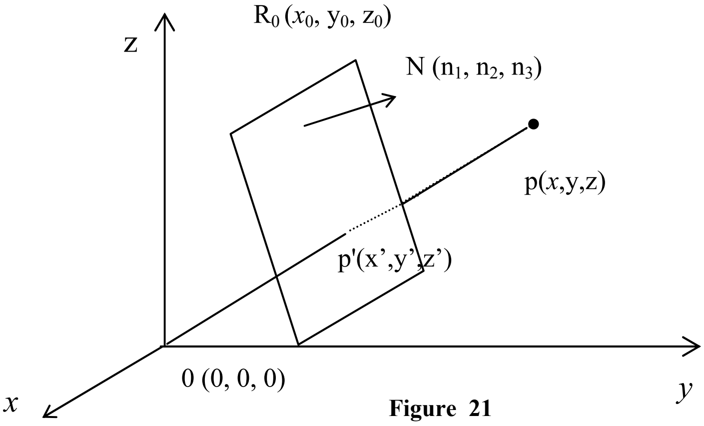

Figure 21

We know that the equation of the projection plane passing through a reference point R_0 and having a normal vector **N**=n_1**I**+n_2**J**+n_3**K** is given by $\mathbf{PR_0.N}$=0, that is

$(x-x_0,y-y_0,z-z_0).(n_1,n_2,n_3)=0$ i.e. $n_1.(x-x_0)+n_2.(y-y_0)+n_3.(z-z_0)=0$ ---------( )

since P'(x',y',z') lies on this plane, thus we have: $n_1.(x'-x_0)+n_2.(y'-y_0)+n_3.(z'-z_0)=0$
After substituting x'=α.x ; y'=α.y ; z'=α.z, we have :

$$\alpha = (n_1.x_0+n_2.y_0+n_3.z_0)/(n_1.x+n_2.y+n_3.z) = d_0/(n_1.x+n_2.y+n_3.z)$$

This projection transformation cannot be represented as a 3x3 matrix transformation. However, by using the homogeneous coordinate representation for 3D, we can write this projection transformation as:

$$P_{per,N,Ro} = \begin{pmatrix} d_0 & 0 & 0 & n_1 \\ 0 & d_0 & 0 & n_2 \\ 0 & 0 & d_0 & n_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, the projected point $P'_h(x',y',z',1)$ of given point $P_h(x, y, z, 1)$ can be obtained as

$$P'h = Ph.\ Pper,N,\ Ro = [x, y, z, 1]\begin{pmatrix} d_0 & 0 & 0 & n_1 \\ 0 & d_0 & 0 & n_2 \\ 0 & 0 & d_0 & n_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{-----------------(16)}$$

$= [d_0.x,\ d_0.y,\ d_0z,\ (n_1.x + n_2.y + n_3.z)]$

Where $d_0 = n_1.x_0 + n_2.y_0 + n_3.\ z_0$.

General Perspective transformation w.r.t. an arbitrary COP

Let the COP is at C(a,b,c), as shown in *Figure 22*.

From *Figure 7*, the vectors **CP** and **CP'** have the same direction. The vector **CP'** is a factor of **CP**, that is **CP'**=α.**CP**

Thus, (x'-a)= α.(x-a) z
(y'-b)= α.(y-b)
(z'-c)= α.(z-c) -------------(17)

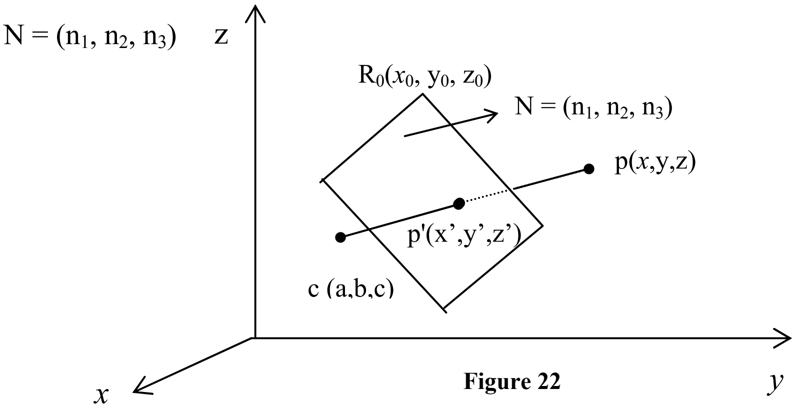

Figure 22

We know that the projection plane passing through a reference point $R_0(x_0,y_0,z_0)$ and having a normal vector N= n1I+n2J+n3K, satisfies the following equation:

n1.(x-x0)+n2.(y-y0)+n3.(z-z0)=0

Since P'(x',y',z') lies on this plane, we have:

n1.(x'-x0)+n2.(y'-y0)+n3.(z'-z0)=0

Substituting the value of x', y' and z', we have:

α= (n1.(x0-a)+n2.(y0-b)+n3.(z0-c))/(n1.(x-a)+n2.(y-b)+n3.(z-c))
=((n1.x0+n2.y0+n3.z0)-(n1.a+n2.b+n3.c))/(n1.(x-a)+n2.(y-b)+n3.(z-c))
=(d0-d1)/(n1.(x-a)+n2.(y-b)+n3.(z-c))
=d/(n1.(x-a)+n2.(y-b)+n3.(z-c))

Here, d=d0-d1= (n1.x0+n2.y0+n3.z0)- (n1.a+n2.b+n3.c) represents perpendicular distance from COP, C to the projection plane.

In order to find out the general perspective transformation matrix, we have to proceed as follows:

Translate COP, C(a,b,c) to the origin. Now, R'0=(x0-a,y0-b,z0-c) becomes the reference point of the translated plane.(but Normal vector will remain same).

Apply the general perspective transformation $P_{per,N,R'o}$

Translate the origin back to C.

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -a & -b & -c & 1 \end{pmatrix} \begin{pmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & 0 \\ n_1 & n_2 & n_3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a & b & c & 1 \end{pmatrix}$$

$$= \begin{pmatrix} d+n_1.a & n_1.b & n_1.c & n_1 \\ n_2.a & d+n_2.b & n_2.c & n_2 \\ n_3.a & n_3.b & d+n_3.c & n_3 \\ -a.d_0 & -b.d_0 & -c.d_0 & -d_1 \end{pmatrix} \qquad \text{------------ (18)}$$

Where d = N.CR' 0 = d0 – d1 = (n1. x0 + n2. Y0 + n3.z0) – (n1.a+n2.b +n3.c)
= n1. (x0 – a) + n2. (y0 – b) + n3. (z0 – c)

And d1 = n1.a + n2.b + n3.c

Example 11: Obtain the perspective transformation onto z = – 2 Plane, where the center of projection is at (0, 0, 18).

Solution: Here centre of projection, C (a, b, c) = (0, 0, 18)

$\therefore$ $(n_1, n_2, n_3) = (0, 0, 1)$

and Reference point R_0 $(x_0, y_0, z_0) = (0, 0, -2)$

$\therefore$ $d_0 = (n_1x_0 + n_2.y_0 + n_3z_0) = -2$

$d_1 = (n_1.a + n_2.b + n_3.\ c) = 18$

we know that the general perspective transformation when cop is not at the origin is given by:

$$\begin{pmatrix} d+n_1.a & n_1.b & n_1.c & n_1 \\ n_2.a & d+n_2.b & n_2.c & n_2 \\ n_3.a & n_3.b & d+n_3.c & n_3 \\ -a.d_0 & -b.d_0 & -c.d_0 & -d_1 \end{pmatrix}$$

$$= \begin{pmatrix} -20 & 0 & 0 & 0 \\ 0 & -20 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 36 & -18 \end{pmatrix} = \begin{pmatrix} -20 & 0 & 0 & 0 \\ 0 & -20 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -2 & 1 \end{pmatrix}$$

Example 12: Find the perspective transformation matrix on to z = 5 plane, when the c.o.p is at origin.

Solution. Since z = 5 is parallel to z = 0 plane, the normal is the same as the unit vecter 'k'.

$\therefore$ $(n_1, n_2, n_3) = (0, 0, 1)$

and the Reference point R0 $(x_0, y_0, z_0) = (0, 0, 5)$

$d_0 = n_1.\ x_0 + n_2.\ y_0 + n_3.\ z_0) = 5$

we know the general perspective transformation, when cop is at origin is given by:

$$\begin{pmatrix} d_0 & 0 & 0 & n_1 \\ 0 & d_0 & 0 & n_2 \\ 0 & 0 & d_0 & n_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

☞ **Check Your Progress 4**

1) Determine the vanishing points for the following perspective transformation matrix:

$$\begin{bmatrix} 8.68 & 5.6 & 0 & 2.8 \\ 0 & 20.5 & 0 & 4.5 \\ 7.0 & 800 & 0 & 2.0 \\ 5.3 & 7.3 & 0 & 3.0 \end{bmatrix}$$

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2) Find the three-point perspective transformation with vanishing points at $V_x = 5$, $V_y = 5$ and $V_z = -5$, for a Given eight vertices of a cube A (0, 0, 1), B (1, 0, 1), C (1, 1, 1) D (0, 1, 1), E (0, 0, 0), F (1, 0, 0), G (1, 1, 0), H (0, 1, 0).

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2.3 SUMMARY

- Projection is basically a transformation (mapping) of 3D objects on 2D screen.
- Projection is broadly categorised into Parallel and Perspective projections depending on whether the rays from the object converge at the COP or not.
- If the distance of COP from the projection plane is finite, then we have Perspective projection. This is called perspective because faraway objects look smaller and nearer objects look bigger.
- When the distance of COP from the projection plane is infinite, then rays from the objects become parallel. This type of projection is called parallel projection.
- Parallel projection can be categorised according to the angle that the direction of projection makes with the projection plane.
- If the direction of projection of rays is perpendicular to the projection plane, we have an Orthographic projection, otherwise an Oblique projection.
- Orthographic (perpendicular) projection shows only one face of a given object, i.e., only two dimensions: length and width, whereas Oblique projection shows all the three dimensions, i.e. length, width and height. Thus, an Oblique projection is one way to show all three dimensions of an object in a single view.
- In Oblique projection the line perpendicular to the projection plane are *foreshortened* (Projected line length is shorter then actual line length) by the direction of projection of rays. The direction of projection of rays determines the amount of foreshortening.
- The change in length of the projected line (due to the direction of projection of rays) is measured in terms of foreshortening factor, f, which is defined as the ratio of the projected length to its true length.

- In Oblique projection, if foreshortening factor f=1, then we have cavalier projection and if f=1/2 then cabinet projection.
- The plane of projection may be perpendicular or may not be perpendicular to the principal axes. If the plane of projection is perpendicular to the principal axes then we have *multiview* projection otherwise *axonometric* projection.
- Depending on the foreshortening factors, we have three different types of Axonometric projections: Isometric (all foreshortening factors are equal), Dimetric (any two foreshortening factors equal) and Trimetric (all foreshortening factors unequal).
- In perspective projection, the parallel lines appear to meet at a point i.e., point at infinity. This point is called vanishing point. A practical example is a long straight railroad track, where two parallel railroad tracks appear to meet at infinity.
- A perspective projection can have at most 3 principal vanishing points (points at infinity w.r.t. x, y, and z-axes, respectively) and at least one principle vanishing point.
- A single point perspective transformation with the COP along any of the coordinate axes yields a single vanishing point, where two parallel lines appear to meet at infinity.
- Two point perspective transformations are obtained by the concatenation of any two one-point perspective transformations. So we can have 3 two-point perspective transformations, namely P_{per-xy}, P_{per-yz}, P_{per-xz} .
- Three point perspective transformations can be obtained by the composition of all the three one-point perspective transformations.

2.4 SOLUTIONS/ANSWERS

Check Your Progress 1

1) Consider a following *Figure m*, where a given line AB is projected to A' B' on a projection plane.

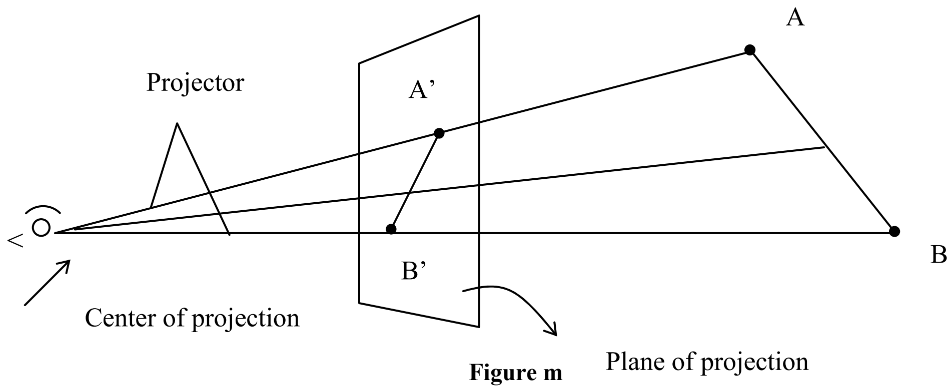

Figure m

a) **Center of projection (cop)**: In case of perspective projection, the rays from an object converge at the finite point, known as center of projection (cop). In *Figure 1*, 0 is the center of projection, where we place our eye to see the projected image on the view plane.

b) **Plane of projection**: Projection is basically a mapping of 3D-object on to 2D-screen. Here 2D-screen, which constitutes the display surface, is known as plane of projection/view plane. That a plane (or display surface), where we are projecting an image of a given 3D-object, is called a plane of projection/view plane. *Figure 1* shows a plane of projection where a given line AB is projected to A'B'.

c) **Projector**: The mapping of 3D-objects on a view plane are formed by projection rays, called the projectors. The intersection of projectors with a view plane form the projected image of a given 3D-object (see *Figure 1*).

d) **Direction of projection**: In case of parallel projection, if the distance of cop from the projection plane is infinity, then all the rays from the object become parallel and will have a direction called "direction of projection". It is denoted by d = (d_1,d_2,d_3), where d_1, d_2 and d_3 make an angle with positive side of *x*, *y* and *z* axes, respectively (see *Figure n*)

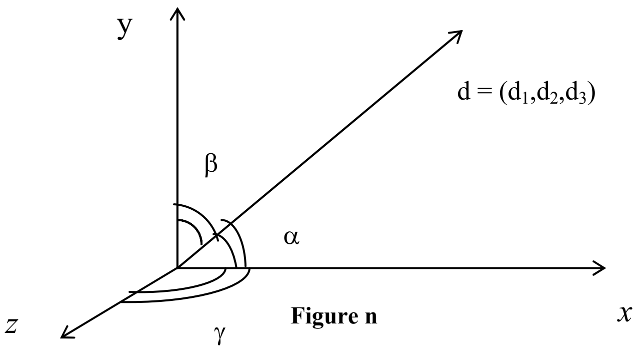

Figure n

The Categorisation of parallel and perspective projection is based on the fact whether coming from the object converge at the cop or not. If the rays coming from the object converges at the centre of projection, then this projection is known as perspective projection, otherwise parallel projection.

Parallel projection can be categorized into orthographic and Oblique projection.

A parallel projection can be categorized according to the angle that the direction of projection $\bar{d}$ makes with the view plane. If $\bar{d}$ is $\perp^r$ to the view plane, then this parallel projection is known as orthographic, otherwise Oblique projection.
Orthographic projection is further subdivided into multiview view plane parallel to the principal axes)

Axonometric projection (view plane not to the principal axes).

Oblique projection is further subdivided into cavalier and canbinet and if f = ½ then cabinet projection.

The *Figure O* shows the Taxonomy of projections:

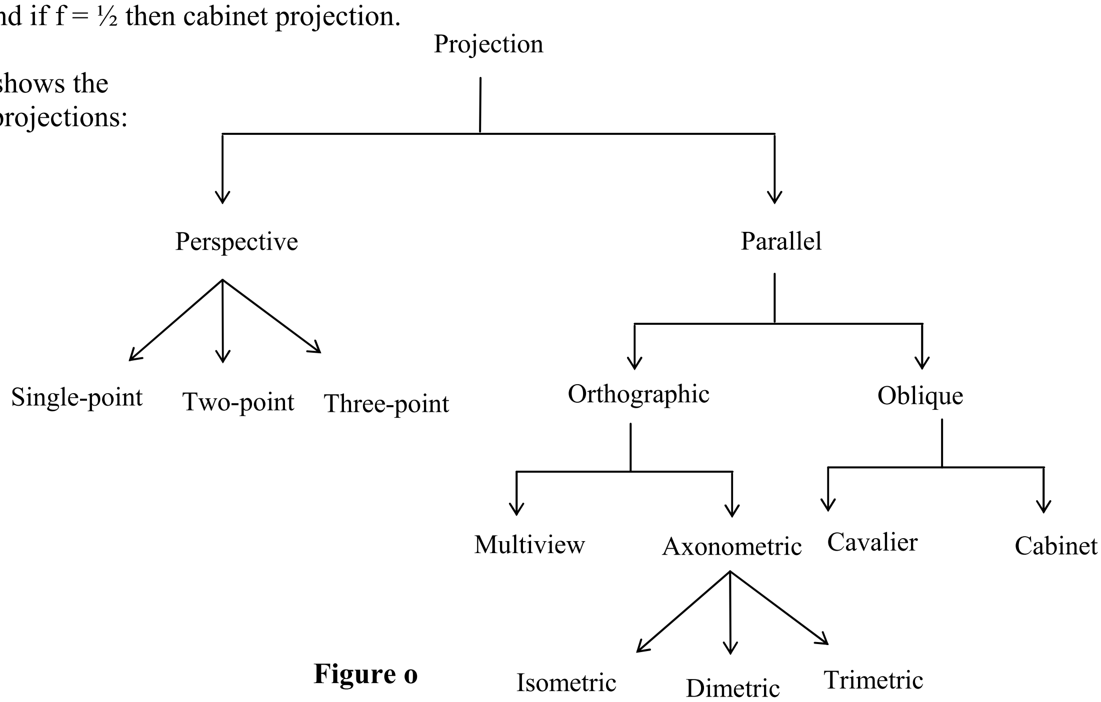

Figure o

3) C

Check Your Progress 2

1) C

2) We know that, the parallel projections can be categorized according to the angle that the direction of projection $\overline{d} = (d_1, d_2, d_3)$ makes with the projection plane. Thus, if direction of projection $\overline{d}$ is $\perp^r$ to the projection plane then we have orthographic projection and if the $\overline{d}$ is not $\perp^r$ to the projection plane then we have oblique projection.

3) The ratio of projected length of a given line to its true length is called the foreshortening factor w.r.t. a given direction.
Let AB is any given line segment
Also assume AB || $\vec{a}$.

Then Under parallel projection, AB is projected to A'B'; The change in the length of projected line is measured in terms of foreshortening factor. f.

$$\therefore\ f = \frac{|A'B'|}{|AB|}$$

Depending on foreshortening factors, we have (3) different types of Axonometric projections:

- Isometric
- Diametric
- Trimetric

When all foreshortening factors along the x-, y- and z-axes are equal, i.e., $f_x = f_y = f_z$, then we have Isometric projection, i.e., the direction of projection makes equal angle with all the positive sides of x, y, and z-axes, respectively.

Similarly, if any two foreshortening factors are equal, i.e., $f_x = f_y$ or $f_y = f_z$ or $f_x = f_z$ then, we have Diametric projection. If all the foreshortening factors are unequal $\overline{d}$ makes unequal angles with *x*, *y*, and *z*-axes/, then we have Trimetric projection.

4) Refer 2. 3. 1. 2 Isometric projection.

5) For orthographic projection, Normal vector N should be parallel to the direction of projection vecter, $\overline{d}$.

i.e. $\overline{d} = k\overline{N}$ where k is a constant.
$(-1, 0, 0) = k(1, 0, -1)$
This is not possible

Hence, the projection plane is not perpendicular to the direction of projection. Hence it is not an orthographic projection.

6) The transformation matrix for cavalier and cabinet projections are given by:

$$P_{cav} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ f.\cos\theta & f.\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \cos 45^\circ & \sin 45^\circ & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{-----1)}$$

$$\&P_{cab}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ f.\cos\theta & f.\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2}.\sin 30° & \frac{1}{2}.\cos 30° & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.43 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{---(2)}$$

The given pyromid can be shown by the following *Figure p*.

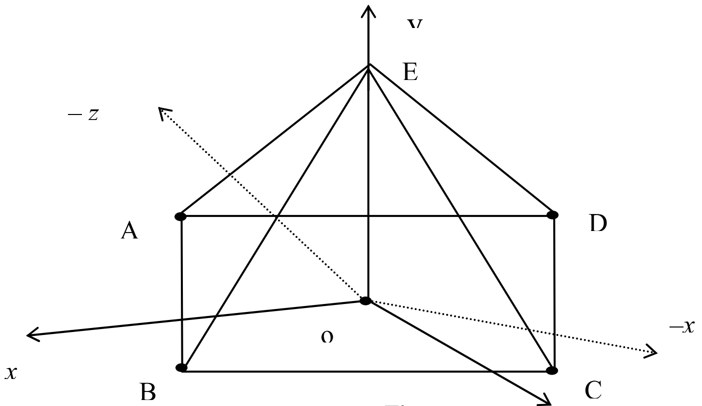

Figure p

The vertices of the pyramid are:

A (2, 0, – 2), B (2, 0, 2), C (– 2, 0, 2)
D (– 2, 0, – 2), E (0, 10, 0)

Using the projection matrices from (1) and (2), we can easily compute the new vertices of the pyramid for cavalier and cabinet projections. (refer Example 4).

Check Your Progress 3

1) Let p (x, y, z) be any point in 3D and the cop is E (0, 0, 0).

The parametric equation of the ray, starting from E and passing through p is:

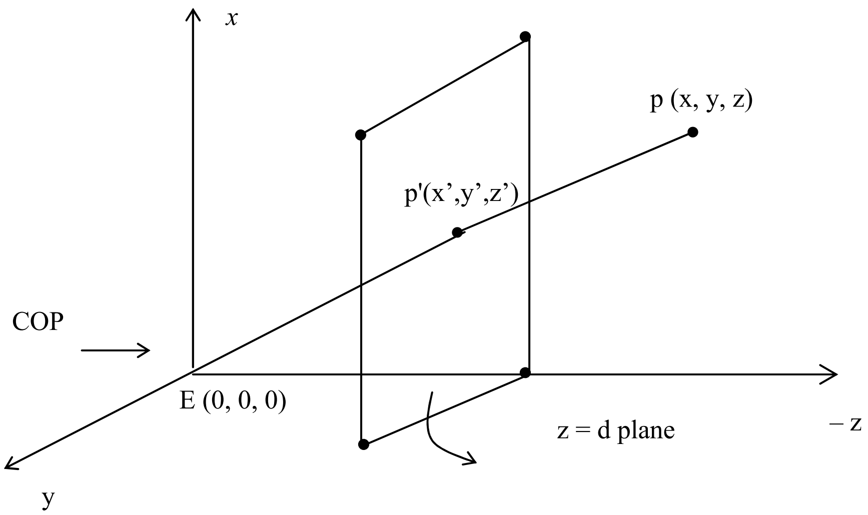

Figure q

E + t (P – E), t > 0
= (0, 0, 0) + t [(x, y, z) – (0, 0, 0)]
= (t.x, t. y, t. z)

For this projected point of p (x, y, z) will be:

$$t.\, z = d$$

$$= \quad t = \frac{d}{z} \text{ must be true.}$$

Hence the projected point of p (x, y, z) will be:

$$P' = (x', y', z') = \left(\frac{d.x}{z}, \frac{d.y}{z}, d\right) \Rightarrow \text{in homogenous Coordinates } \left(\frac{dx}{z}, \frac{dy}{z}, d, 1\right)$$

$$= (dx, dy, dz, z)$$

In matrix form:

$$(x', y', z', 1) = (x, y, z, 1) \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2) Since the cube is first translated by –0.5 units in the *x* and *y*-directions, to get the centred cube on the *z*-axis.

The transformation matrix for translation is:

$$[T_{x,y}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -0.5 & -0.5 & 0 & 1 \end{bmatrix} \qquad \text{----------(1)}$$

A single-point perspective transformation onto the z = 0 plane is given by:

$$P_{per,z} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/d \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{----------(2)}$$

It has a center of projection on the z-axis: at $d = -10 \Rightarrow \frac{1}{d} = -0.1$

From equation (2)

$$P_{per,z} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The resulting transformation can be obtained as:

$$[T] = [T_{x,y}].\, [P_{per,z}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.1 \\ -0.5 & -0.5 & 0 & 1 \end{bmatrix}$$

Thus, the projected points of the centred cube V = [ABCDEFGH] will be:

$$[V'] = [V].\,[T] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.1 \\ -0.5 & -0.5 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.5 & -0.5 & 0 & 0.9 \\ 0.5 & -0.5 & 0 & 0.9 \\ 0.5 & 0.5 & 0 & 0.9 \\ -0.5 & 0.5 & 0 & 0.9 \\ -0.5 & -0.5 & 0 & 1 \\ 0.5 & -0.5 & 0 & 1 \\ 0.5 & 0.5 & 0 & 1 \\ -0.5 & 0.5 & 0 & 1 \end{bmatrix} = \begin{array}{c} A' \\ B' \\ C' \\ D' \\ E' \\ F' \\ G' \\ H' \end{array} \begin{bmatrix} -0.56 & -0.56 & 0 & 1 \\ 0.56 & -0.56 & 0 & 1 \\ 0.56 & 0.56 & 0 & 1 \\ -0.56 & -0.56 & 0 & 1 \\ -0.5 & -0.5 & 0 & 1 \\ 0.5 & -0.5 & 0 & 1 \\ 0.5 & 0.5 & 0 & 1 \\ -0.5 & 0.5 & 0 & 1 \end{bmatrix}$$

3) A unit cube is placed at the origin such that its 3-edges are lying along the x,y, and z-axes. The cube is rotated about the y-axis by 30°. Obtain the perspective projection of the cube viewed from (80, 0, 60) on the z= 0 plane.

3) Rotation of a cube by 30° along y-axis,

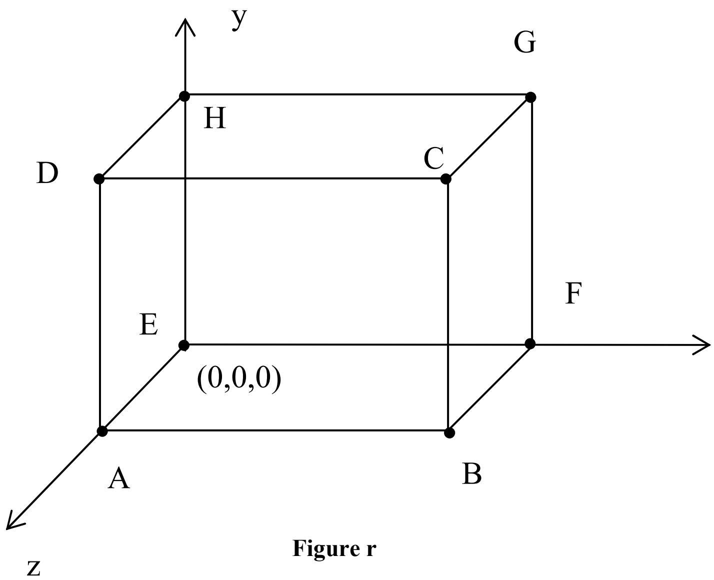

Figure r

$$[R_y]_{30°} = \begin{bmatrix} \cos 30° & 0 & -\sin 30° & 0 \\ 0 & 1 & 0 & 0 \\ \sin 30° & 0 & \cos 30° & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{3}/2 & 0 & -1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 1/2 & 0 & \sqrt{3}/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.86 & 0 & -0.5 & 0 \\ 0 & 1 & 0 & 0 \\ 0.5 & 0 & 8.6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let p (x, y, z) be any point of a cube in a space and p' (x', y', z') is its projected point onto z = 0 plane.

The pametric equation of a line, starting from E (80, 0, 60) and passing through P (x, y, z) is:

E + t (P – E), 0 < t ∞.
= (80, 0, 60) + t [(x, y, z) – (80, 0, 60)]
= (80, 0, 60) + t[(x – 80), y, (z – 60)]
= [t (x – 80) + 80, t.y, t. (z – 60) + 60]

Assume point P' can be obtained, when t = t*
⇒ P' = (x', y', z') = [t* (x – 80) + 80, t*.y, t* (z – 60) + 60]
Since point p' lies on z = 0 plane, so

$t^* (z - 60) + 60 = 0 \Rightarrow t^* = \dfrac{-60}{z-60}$

$\Rightarrow p' = (x', y', z') = \left(\dfrac{-60.x + 80.z}{z-60}, \dfrac{-60.y}{z-60}, 0\right)$

In Homogeneous coordinates system:

$$P' (x', y', z', 1) = \left(\frac{-60.x + 80.z}{z-60}, \frac{-60.y}{z-60}, 0, 1\right)$$
$$= (-60.\ x + 80.\ z, -60.\ Y, 0, z - 60)$$

In Matrix form:

$$(x', y', z', 1) = (x, y, z, 1). \begin{bmatrix} -60 & 0 & 0 & 0 \\ 0 & -60 & 0 & 0 \\ 80 & 0 & 0 & 1 \\ 0 & 0 & 0 & -60 \end{bmatrix} \text{------------(1)}$$

$$P'_n = P_n.\ P_{par,\ z} \qquad \text{-------------(2)}$$

Since a given cube is rotated about y-axis by 30°, so the final projected point p' (of a cube on z = 0 plane) can be obtained as follows:

$$P'_n = P_n.\ [R_y]_{30°}.\ P_{par,\ z}$$

$$(x', y', z', 1) = (x, y, z, 1)\ . \begin{bmatrix} 0.86 & 0 & -0.5 & 0 \\ 0 & 1 & 0 & 0 \\ 0.5 & 0 & 0.86 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} -60 & 0 & 0 & 0 \\ 0 & -60 & 0 & 0 \\ 80 & 0 & 0 & 1 \\ 0 & 0 & 0 & -60 \end{bmatrix}$$

$$(x', y', z', 1) = (x, y, z, 1) \begin{bmatrix} 91.9 & 0 & 0 & -0.5 \\ 0 & -60 & 0 & 0 \\ 38.8 & 0 & 0 & 0.86 \\ 0 & 0 & 0 & -60 \end{bmatrix} \text{-------------(3)}$$

$$P''n = P.\ P_{par}, z, 30°$$

This equation (3) is the required perspective transformation. Which gives a coordinates of a projected point P' (x', y', z') onto the z = 0 plane, when a point P (x, y, z) is viewed from E (80, 0, 60).

Thus, all the projected points of a given cube can be obtained as follows:

$$P' = V.\ Ppar,\ z,\ 30° = \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{array} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 90.9 & 0 & 0 & -0.5 \\ 0 & -60 & 0 & 0 \\ 38.8 & 0 & 0 & 0.86 \\ 0 & 0 & 0 & -60 \end{bmatrix}$$

$$= \begin{array}{c} A' \\ B' \\ C' \\ D' \\ E' \\ F' \\ G' \\ H' \end{array} \begin{bmatrix} 38.8 & 0 & 0 & -59.14 \\ 129.7 & 0 & 0 & -59.64 \\ 129.7 & -60 & 0 & -59.64 \\ 38.8 & -60 & 0 & -60.86 \\ 0 & 0 & 0 & -60.0 \\ 90.9 & 0 & 0 & -60.5 \\ 90.9 & -60 & 0 & -60.5 \\ 0 & -60 & 0 & -60.0 \end{bmatrix} = \begin{array}{c} A' \\ B' \\ C' \\ D' \\ E' \\ F' \\ G' \\ H' \end{array} \begin{bmatrix} -0.72 & 0 & 0 & 1 \\ -2.17 & 0 & 0 & 1 \\ -2.17 & 1.01 & 0 & 1 \\ -0.64 & 0.99 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1.50 & 0 & 0 & 1 \\ -1.50 & 0.99 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Hence, A' = (– 0.72, 0, 0), B' = (– 2.17, 0, 0), C' = (– 2.17, 1.01, 0)
D' = (– 0.64, 0.99, 0), E' = (0, 0, 0), F' = (– 1.5, 0, 0)
G' = (– 1.50, 0.99, 0) and H' = (0, 1, 0).

Check Your Progress 4

1) The given perspective transformation matrix can be written as:

From Rows one, two and three from equation matrix (I), the vanishing point w.r.t. x, y and z axis, will be:

$C_x = (3.1, 2.0, 0)$
$C_y = (0, 4.56, 0)$
$C_z = (3.5, 4.0, 0)$

2) From the given V.P., we can obtain the corresponding center of projections. Since vanishing points: $V_x = 5$, $V_y = 5$ and $V_z = -5$, hence center of projections is at:

$$C_x = -5,\ C_y = -5 \text{ and } C_z = 5$$

$$\therefore 1/d_1 = \frac{1}{5} = 0.2,\ \frac{1}{d_2} = \frac{1}{5} = 0.2 \text{ and } \frac{1}{d_3} = \frac{-1}{5} = -0.2$$

Hence, the 3 – point perspective transformation is:

$$P_{per-xyz} = \begin{bmatrix} 1 & 0 & 0 & 0.2 \\ 0 & 1 & 0 & 0.2 \\ 0 & 0 & 1 & -0.2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{------------(I)}$$

Thus by multiplying v = [ABCDEFGH] with projection matrix (I), we can obtain the transformed vertices of a given cube.